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Continuous Optimization

# Quadratic convex reformulation for quadratic programming with linear on–off constraints

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ABSTRACT

In production planning and resource allocation problems, we often encounter a situation where a constraint can be relaxed or removed if new resources are added. Such constraints are termed on–off constraints. We study the quadratic programming problem with such on–off constraints, which is in general NP-hard. As the problem size grows, branch-and-bound algorithms for the standard formulation of this problem often require a lot of computing time because the lower bound from the continuous relaxation is in general quite loose. We generalize the quadratic convex reformulation (QCR) approach in the literature to derive a new reformulation that can be solved by standard mixed-integer quadratic programming (MIQP) solvers with less computing time when the problem size becomes large. While the conventional QCR approach utilizes a quadratic function that vanishes on the entire feasible region, the approach proposed in our paper utilizes a quadratic function that only vanishes on the set of optimal solutions. We prove that the continuous relaxation of our new reformulation is at least as tight as that of the best reformulation in the literature. Our computational tests verify the effectiveness of our new approach.



## 1. Introduction

In this paper we study the following mixed-integer quadratic programming (MIQP) problem:

$$\begin{aligned}
(\mathrm{P})\quad & \min_{x,y}\ f \triangleq x^T Q x + c^T x + h^T y \\
& \text{s.t.}\quad a_i - b_i y_i \le A_i x,\ \ i = 1,\dots,m, && (1)\\
& \qquad Bx + Dy \le d, && (2)\\
& \qquad y \in \{0,1\}^m, x \in \mathbb{R}^n, && (3)
\end{aligned}$$

where $Q \in \mathbb{R}^{n\times n}$ is symmetric and positive definite, $c \in \mathbb{R}^n$, $h \in \mathbb{R}^m$, $A = [A_1^T, \dots, A_m^T]^T \in \mathbb{R}^{m\times n}$, $a = (a_1, \dots, a_m)^T \in \mathbb{R}^m$, $b = (b_1, \dots, b_m)^T \in \mathbb{R}^m_+$, $B \in \mathbb{R}^{M\times n}$, $D \in \mathbb{R}^{M\times m}$, and $d \in \mathbb{R}^M$. The constraint (1) is introduced to represent the situation where we can relax the constraint by adding a resource $b_i$ while incurring a fixed cost of $h_i$. It can also be interpreted as an linear "on/off" constraint where the constraint is activated (on) when the boolean variable $y_i$ is zero and is not activated (off) when $y_i$ is one. This constraint has many applications such as in production planning (Bestuzheva, Hijazi, & Coffrin, 2016), facility location (Cornnejols, Fisher, & Nemhauser, 1977) and supervised classification (Belotti et al., 2016). Problem (P) is NP-hard since its special case with $Q = 0$ is already NP-hard (Benati & Rizzi, 2007). Our formulation also includes the probabilistically-constrained quadratic programming problem with finite distributions as a special case (Zheng, Sun, Li, & Cui, 2012).

Problem (P) can be solved by existing MIQP solvers, such as CPLEX and Gurobi, within a branch-and-bound framework using continuous relaxations. However, when (P) is directly solved by standard MIQP solvers, the performance is not satisfactory, as the continuous relaxation of formulation (P) usually yields a loose lower bound[1]. Efficient MIQP reformulations to problem (P) were derived in Zheng et al. (2012) and Hsia et al. (2014) via a family of auxiliary semi-definite programming (SDP) problems.

The purpose of this paper is to reformulate problem (P) to an equivalent MIQP reformulation that is easier to solve as the problem size grows. An MIQP reformulation can then be plugged into and solved by any off-the-shelf softwares such as CPLEX and

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[1] Hsia, Wu, and Li (2014) provided an example. They also proved that their new reformulation has a tighter continuous relaxation lower bound than that from the standard formulation.

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Gurobi. In general, the effectiveness of a reformulation depends mainly on two factors. The first factor is the quality of the lower bound provided by the continuous relaxation of the reformulation. Tighter lower bounds usually lead to a smaller-sized branch-and-bound tree and less searching time. The second factor is the time needed to solve the continuous relaxations of the reformulation. Even if the continuous relaxation has a tight lower bound, if the amount of time needed to compute the lower bound is large, the reformulation is not effective. For example, if the relaxation problem is not convex, it could be very difficult to obtain the lower bound.

We first follow the work of Zheng et al. (2012) and Hsia et al. (2014) and show that their reformulations can be simplified with a reduced number of decision variables. We prove that these reduced reformulations will be at least as tight as their original reformulations[2].

Next, we generalize the quadratic convex reformulation (QCR) approach in the literature (See, e.g., Billionnet, Elloumi, & Plateau, 2008; Billionnet, Elloumi, & Plateau, 2009; Plateau, 2006) to derive a new reformulation that is at least as tight as the reduced reformulation of Zheng et al. (2012). Also, our reformulation strictly dominates the reduced reformulation of Hsia et al. (2014). The QCR approach in the literature appends to the original objective function a quadratic function that vanishes on the feasible region and is confined to situations with equality constraints. While the QCR approach requires the added quadratic function to vanish on the entire feasible region, the approach proposed in our paper only requires the added quadratic function to vanish on the set of optimal solutions. With this significant advantage, we can extend the QCR approach to more general situations with inequality constraints.

We also conduct numerical comparison between the reduced reformulation of Zheng et al. (2012) and our new reformulation. Preliminary experiments show that our new reformulation is more effective when solved in standard MIQP solvers such as CPLEX.

Note that on/off constraints are also defined using the following form in the literature,

$$x \in \Gamma_0 \cup \Gamma_1,$$

$$\Gamma_0 = \{(x, y) \in \mathbb{R}^n \times \{0, 1\} \mid y = 0,\ l^0 \le x \le u^0\},$$

$$\Gamma_1 = \{(x, y) \in \mathbb{R}^n \times \{0, 1\} \mid y = 1,\ g(x) \le 0,\ l^1 \le x \le u^1\}.$$

A compact characterization of the convex hull was proposed in Günlük and Linderoth (2010) when the set $\Gamma_0$ is a singleton. The result was then generalized by Hijazi, Bonami, Cornuéjols, and Ouorou (2012) to cases where $\Gamma_0$ is a hyper-rectangle and the constraint function $g(x)$ is isotone. Convex quadratic relaxations were derived in Hijazi, Coffrin, and Van Hentenryck (2013) for power system problems with nonlinear on/off constraints. A detailed literature review can be found in Bonami, Lodi, Tramontani, and Wiese (2015) for mathematical programming with such on/off constraints.

The remaining of the paper is organized as follows. In Section 2 we review the related reformulations in the literature. In Section 3 we reduce the existing reformulations discussed in Section 2 to reformulations with fewer decision variables. We prove that the reduced reformulations are at least as tight as the original reformulations in Zheng et al. (2012) and Hsia et al. (2014). In Section 4 we review the QCR approach in the literature and show how to generalize the state-of-the-art of QCR to obtain new reformulations. We show that the construction of the best reformulation reduces to an SDP problem. In Section 5 we establish the tightness of our new reformulation. In Section 6 we conduct numerical tests to demonstrate that the performance of our new reformulation is better than the existing results when solved in standard MIQP solvers. Finally we conclude the paper in Section 7.

*Notation:* Throughout this paper, we denote by v(•) the optimal value of problem (•), and $\mathbb{R}^n_+$ the nonnegative orthant of $\mathbb{R}^n$. For any $a \in \mathbb{R}^n$, we denote by $\mathrm{Diag}(a) = \mathrm{Diag}(a_1, \dots, a_n)$ the diagonal matrix with $a_i$ being its $i$th diagonal element. We denote by $e$ the all-one vector. Denote the continuous relaxation problem of (•) by $(\overline{\bullet})$, which relaxes the binary constraint $y \in \{0, 1\}^m$ to the following linear constraints:

$$0 \le y_i \le 1,\ i = 1, \dots, m. \tag{4}$$

Denote by (P($\theta$)) a problem formulation that is parameterized by $\theta$. Denote by (P($u, v, w$)) a problem formulation that is parameterized by $u, v$, and $w$. Denote by $q(\cdot,\ \cdot,\ \cdot)$ a function on three variables. We list here the main problem formulations studied in this paper to facilitate a clear reference for readers:

- Section 2:
  - (P1($\theta$)): Reformulation of (P) derived in Zheng et al. (2012).
  - ($\overline{\mathrm{P1}}(\theta)$): The continuous relaxation of (P1($\theta$)).
  - (P2($\theta$)): Reformulation of (P) derived in Hsia et al. (2014).
  - ($\overline{\mathrm{P2}}(\theta)$): The continuous relaxation of (P2($\theta$)).
- Section 3:
  - (P1R($\theta$)): Reduced reformulation of (P1($\theta$)).
  - ($\overline{\mathrm{P1R}}(\theta)$): The continuous relaxation of (P1R($\theta$)).
  - (P2R($\theta$)): Reduced reformulation of (P2($\theta$)).
  - ($\overline{\mathrm{P2R}}(\theta)$): The continuous relaxation of (P2R($\theta$)).
- Section 4:
  - (P($q$)): Our new reformulation for (P) parameterized by a quadratic function $q$.
  - ($\mathrm{P}_q(u, v, w)$): Our new reformulation for (P) parameterized by $u, v$ and $w$.
  - ($\overline{\mathrm{P}}_q(u, v, w)$): The continuous relaxation of ($\mathrm{P}_q(u, v, w)$).
- Section 5:
  - ($\mathrm{P'}_q(u, v, w)$)): An alternative form of our new reformulation ($\mathrm{P}_q(u, v, w)$).
  - ($\overline{\mathrm{P'}}_q(u, v, w)$): The continuous relaxation of ($\mathrm{P'}_q(u, v, w)$).

## 2. Related reformulations

In this section, we review a few related reformulations and reformulation technique in the literature. Zheng et al. (2012) derived the following parameterized reformulation of (P):

$$
\begin{aligned}
(\mathrm{P1}(\theta)) \quad \min_{x,y,z,w,\phi} \quad & f_1 \triangleq x^T(Q - A^T\mathrm{Diag}(\theta)A)x + c^Tx + h^Ty \\
& + \sum_{i=1}^{m} \theta_i(w_i^2 + \phi_i - a_i^2 y_i) \\
\text{s.t.} \quad & A_i x = w_i + z_i - y_i a_i,\ i = 1, \dots, m, && (5)\\
& (a_i - b_i)y_i \le z_i \le a_i y_i,\ i = 1, \dots, m, && (6)\\
& a_i \le w_i,\ i = 1, \dots, m, && (7)\\
& z_i^2 \le \phi_i y_i,\ \phi_i \ge 0,\ i = 1, \dots, m, && (8)\\
& w = (w_1, \dots, w_m)^T \in \mathbb{R}^m, && (9)\\
& z = (z_1, \dots, z_m)^T \in \mathbb{R}^m, && (10)\\
& \phi = (\phi_1, \dots, \phi_m)^T \in \mathbb{R}^m, && (11)\\
& (2), (3),
\end{aligned}
$$

where $\theta \in \mathbb{R}^m$ is a parameter vector. They searched for the best parameter $\theta$ by solving the following problem:

$$\max_{\theta \in \mathbb{R}^m} \{\mathrm{v}(\overline{\mathrm{P1}}(\theta)) \mid f_1 \text{ is a convex function.}\}, \tag{12}$$

which can be solved via an equivalent SDP formulation.

[2] We say a reformulation is tight if the continuous relaxation of this reformulation provides a tight lower bound.

Hsia et al. (2014) derived the following parameterized reformulation of (P):

$$
\begin{aligned}
(\mathrm{P2}(\theta)) \quad \min_{x,y,z,w} \quad & f_2 \triangleq x^T(Q - A^T\mathrm{Diag}(\theta)A)x + c^Tx + h^Ty \\
& + \sum_{i=1}^{m} \theta_i(w_i^2 + z_i^2 - a_i^2 y_i) \\
\text{s.t.} \quad & (2), (3), (9), (10), (5), (6), (7),
\end{aligned}
$$

where $\theta \in \mathbb{R}^m$ is a parameter vector. They searched for the best parameter $\theta$ by solving the following problem:

$$
\max_{\theta \in \mathbb{R}^m} \{ \mathrm{v}(\overline{\mathrm{P2}}(\theta)) \mid f_2 \text{ is a convex function.} \}, \tag{13}
$$

which can also be solved via an equivalent SDP formulation.

Next, we review the QCR approach in the literature for solving integer or mixed-integer quadratic programming problems. The conventional QCR approach possesses the following characteristics:

- **(a)** An additional function that is zero in the entire feasible region is added to the original objective function to form a new objective function.
- **(b)** The newly formed objective function is convex, so that the continuous relaxation of the new reformulation is convex and thus can be solved efficiently.
- **(c)** The new reformulation is configured in a way that its continuous relaxation is tighter than that of the original formulation, so that the new reformulation can be solved by branch-and-bound methods more efficiently.

Hammer and Rubin (1970) pioneered the QCR approach for solving the following binary quadratic programs:

$$
\min_x \left\{ x^TQx + c^Tx \;\middle|\; Ax = d,\; x_i \in \{0,1\},\; i = 1,\dots,n \right\}, \tag{14}
$$

where $Q$ is indefinite. In their proposed QCR, Hammer and Rubin (1970) added to the objective function a term $\sum_i u(x_i^2 - x_i)$, where $u$ is a scalar and is chosen to be the negative value of the smallest eigenvalue of $Q$. Billionnet and Elloumi (2007) improved this method by adding the term $\sum_i u_i(x_i^2 - x_i)$ with $u_i$ being the optimal dual variables of a certain semi-definite program (SDP). Plateau (2006) and Billionnet et al. (2008, 2009) also utilized the equality $Ax = d$ in QCR and added the term $\sum_i u_i(x_i^2 - x_i) + \theta(Ax - d)^T(Ax - d)$ to the objective, where $u$ and $\theta$ are chosen to be the dual variables of an enlarged SDP program (Here $\theta$ is a scalar[3]). Ahlatçıoğlu et al. (2012) proposed to combine QCR and the convex hull relaxation to solve problem (14). Using binary expansion, the QCR approach was extended to general mixed-integer quadratic programs in Billionnet, Elloumi, and Lambert (2012), Billionnet, Elloumi, and Lambert (2013), and Billionnet, Elloumi, and Lambert (2015).

## 3. Dimension reduction

In this section, we show that the reformulations reported in the previous section can be reduced to reformulations with fewer variables. Also, we prove that the reduced reformulations are at least as tight as their original reformulations.

The $w$ vector in problem (P1($\theta$)) can be eliminated using the equality constraint (5), resulting in the following reformulation with a reduced number of decision variables,

$$
\begin{aligned}
(\mathrm{P1R}(\theta)) \quad \min_{x,y,z,\phi} \quad & f_{1r} \triangleq f + \sum_{i=1}^{m} \theta_i(a_i^2y_i^2 + z_i^2 + 2a_iA_ixy_i - 2A_ixz_i \\
& - 2a_iy_iz_i + \phi_i - a_i^2y_i) \\
\text{s.t.} \quad & a_i \le A_ix - z_i + a_iy_i,\; i = 1,\dots,m, \qquad (15) \\
& (2), (3), (10), (11), (6), (8).
\end{aligned}
$$

**Example 1.** Consider the following example problem:

$$
\begin{aligned}
\min_{x,y} \quad & x^T\begin{bmatrix} 7 & 3 \\ 3 & 6 \end{bmatrix}x + \begin{bmatrix} -7 \\ -5 \end{bmatrix}^T x + 40y \\
\text{s.t.} \quad & 5 - 30y \le -8x_1 + x_2, \\
& -5x_1 + 2x_2 - 6y \le 3, \\
& y \in \{0,1\}, x = (x_1, x_2)^T \in \mathbb{R}^2.
\end{aligned}
$$

For given $\theta$, the corresponding reformulation (P1R($\theta$)) is:

$$
\begin{aligned}
\min_{x,y,z,\phi} \quad & x^T\begin{bmatrix} 7 & 3 \\ 3 & 6 \end{bmatrix}x + \begin{bmatrix} -7 \\ -5 \end{bmatrix}^T x + 40y \\
& + \theta(5^2y^2 + z^2 + 2\cdot 5(-8x_1 + x_2)y - 2(-8x_1 + x_2)z \\
& - 2\cdot 5yz + \phi - 5^2y) \\
\text{s.t.} \quad & 5 \le -8x_1 + x_2 - z + 5y, \\
& -5x_1 + 2x_2 - 6y \le 3, \\
& (5 - 30)y \le z \le 5y, \\
& z^2 \le \phi y,\; \phi \ge 0, \\
& y \in \{0,1\}, x = (x_1, x_2)^T \in \mathbb{R}^2, z, \phi \in \mathbb{R}.
\end{aligned}
$$

The best parameters $\theta$ for (P1R($\theta$)) can be found by solving the following problem:

$$
\max_{\theta \in \mathbb{R}^m} \{ \mathrm{v}(\overline{\mathrm{P1R}}(\theta)) \mid f_{1r} \text{ is a convex function.} \} \tag{16}
$$

Note that $f_{1r}$ is convex if and only if the following condition on its Hessian holds:

$$
Q_{1r} \triangleq \begin{pmatrix} Q & A^T\mathrm{Diag}(a)\mathrm{Diag}(\theta) & -A^T\mathrm{Diag}(\theta) \\ \mathrm{Diag}(\theta)\mathrm{Diag}(a)A & \mathrm{Diag}(a)\mathrm{Diag}(\theta)\mathrm{Diag}(a) & -\mathrm{Diag}(a)\mathrm{Diag}(\theta) \\ -\mathrm{Diag}(\theta)A & -\mathrm{Diag}(a)\mathrm{Diag}(\theta) & \mathrm{Diag}(\theta) \end{pmatrix} \succeq 0. \tag{17}
$$

The tightness of the reduced reformulation (P1R($\theta$)) is guaranteed by the following proposition.

**Proposition 1.** v*(16)* ≥ v*(12)*.

**Proof.** Assume that $\theta$ is feasible for problem (12), then $\theta \ge 0$ and the Hessian matrix of the function $f_1$ is positive semidefinite. This means that the quadratic part $x^T(Q - A^T\mathrm{Diag}(\theta)A)x$ is convex in $x$. If $w_i^2$ is replaced by $(A_ix - z_i + y_ia_i)^2$, then the function $f_1$ becomes $f_{1r}$. Because the quadratic parts $x^T(Q - A^T\mathrm{Diag}(\theta)A)x$ and $\theta_i(A_ix - z_i + y_ia_i)^2$ are all convex functions, their summation is also convex and the Hessian matrix of $f_{1r}$ remains positive semidefinite. Thus $\theta$ is also feasible for problem (16). For the same $\theta$, we have $\mathrm{v}(\overline{\mathrm{P1}}(\theta)) = \mathrm{v}(\overline{\mathrm{P1R}}(\theta))$, thus v(16) ≥ v(12). □

**Proposition 2.** *If the Slater condition holds for the set*

$$
\{(x, y, z, \phi) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \mid (2), (4), (6), (8), (15)\},
$$

*then problem (16) is equivalent to the following SDP problem:*

$$
\max_{\eta,\beta,\mu,\sigma,\pi,u,\lambda,w,\tau,\theta} \quad \tau \tag{18}
$$

[3] In Billionnet et al. (2008) and Billionnet et al. (2009), a more complex form of $\sum_{r=1}^{M}(\sum_{i=1}^{n}\theta_{ri}x_i)(\sum_{j=1}^{n}A_{rj}x_j - b_r)$ was also considered. It has been shown in Faye and Roupin (2007) that the effect of adding this more complex form to the objective function is the same as adding $\theta(Ax - d)^T(Ax - d)$.

$$\text{s.t.}\quad \begin{pmatrix} Q_{1r} & 0.5G \\ 0.5G^T & -\eta^T d+\beta^T a-e^T\pi-\tau \end{pmatrix}\succeq 0,$$

$$\eta\in\mathbb{R}^M_+,\ \beta\in\mathbb{R}^m_+,\ \mu\in\mathbb{R}^m_+,\ \sigma\in\mathbb{R}^m_+,\ \pi\in\mathbb{R}^m_+, \tag{19}$$

$$\begin{pmatrix} u_i & w_i \\ w_i & \lambda_i \end{pmatrix}\succeq 0,\ u_i,w_i,\lambda_i\in\mathbb{R},\ i=1,\dots,m, \tag{20}$$

$$u=\theta,\ \theta\in\mathbb{R}^m,\ \tau\in\mathbb{R}, \tag{21}$$

*where*

$$G=\begin{pmatrix} c+B^T\eta-A^T\beta \\ h-\mathrm{Diag}(a)\mathrm{Diag}(a)\theta+D^T\eta-\mathrm{Diag}(a)\beta \\ +\mathrm{Diag}(a-b)\mu-\mathrm{Diag}(a)\sigma+\pi-\lambda \\ \beta-\mu+\sigma-2w \end{pmatrix}.$$

**Proof.** We first express $(\overline{\mathrm{P1R}}(\theta))$ by its dual form. Note the following equivalence in the constraints:

$$\{z_i^2\le\phi_i y_i,\ \phi_i\ge 0,\ y_i\ge 0,\ i=1,\dots,m\}\iff\begin{pmatrix}\phi_i & z_i\\ z_i & y_i\end{pmatrix}\succeq 0.$$

Associate the following multipliers to the constraints in $(\overline{\mathrm{P1R}}(\theta))$:

- $\eta\in\mathbb{R}^M_+$ for $Bx+Dy\le d$;
- $\beta_i\in\mathbb{R}_+$ for $a_i\le A_ix-z_i+a_iy_i,\ i=1,\dots,m$;
- $\mu_i$ and $\sigma_i\in\mathbb{R}_+$ for $(a_i-b_i)y_i\le z_i$ and $z_i\le a_iy_i$, respectively, $i=1,\dots,m$;
- $\pi_i\in\mathbb{R}_+$ for $y_i\le 1$, $i=1,\dots,m$;
- $\begin{pmatrix}u_i & w_i\\ w_i & \lambda_i\end{pmatrix}\succeq 0,\ u_i,w_i,\lambda_i\in\mathbb{R}$ for $\begin{pmatrix}\phi_i & z_i\\ z_i & y_i\end{pmatrix}\succeq 0$, respectively, $i=1,\dots,m$.

Let $\mu=(\mu_1,\dots,\mu_m)^T$, $\sigma=(\sigma_1,\dots,\sigma_m)^T$, $\beta=(\beta_1,\dots,\beta_m)^T$, $\lambda=(\lambda_1,\dots,\lambda_m)^T$, $u=(u_1,\dots,u_m)^T$, $w=(w_1,\dots,w_m)^T$ and $\pi=(\pi_1,\dots,\pi_m)^T$. The Lagrangian function of $(\overline{\mathrm{P1R}}(\theta))$ is then given by

$$\begin{aligned}
L(\cdot)&=L(x,y,z,\phi;\eta,\beta,\mu,\sigma,\pi,u,w,\lambda;\theta)\\
&=f_{1r}+\eta^T(Bx+Dy-d)+\sum_{i=1}^m\beta_i(a_i-a_iy_i-A_ix+z_i)\\
&\quad+\sum_{i=1}^m\mu_i((a_i-b_i)y_i-z_i)+\sum_{i=1}^m\sigma_i(z_i-a_iy_i)+\sum_{i=1}^m\pi_i(y_i-1)\\
&\quad-\sum_{i=1}^m(u_i\phi_i+2w_iz_i+\lambda_iy_i)\\
&=(x^T,y^T,z^T)Q_{1r}(x^T,y^T,z^T)^T\\
&\quad+(c+B^T\eta-A^T\beta)^Tx\\
&\quad+(h-\mathrm{Diag}(a)\mathrm{Diag}(a)\theta+D^T\eta-\mathrm{Diag}(a)\beta+\mathrm{Diag}(a-b)\mu\\
&\quad-\mathrm{Diag}(a)\sigma+\pi-\lambda)^Ty\\
&\quad+(\beta-\mu+\sigma-2w)^Tz\\
&\quad+(\theta-u)^T\phi\\
&\quad-\eta^Td+\beta^Ta-e^T\pi.
\end{aligned}$$

Under the assumed Slater condition, problem (16) is equivalent to the following problem by Lagrangian duality:

$$\max_{\theta\in\mathbb{R}^m,(17),(19),(20)}\left\{\min_{(x,y,z,\phi)\in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m}L(\cdot)\right\}. \tag{22}$$

If (17) does not hold, then the inner minimization of problem (22) is an unconstrained non-convex quadratic minimization problem and its optimal value is $-\infty$. Thus, problem (22) can be simplified to:

$$\max_{\theta\in\mathbb{R}^m,(19),(20)}\left\{\min_{(x,y,z,\phi)\in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m}L(\cdot)\right\}. \tag{23}$$

Following Corollary 4.2 in Lemaréchal and Oustry (1999), problem (23) is equivalent to the SDP problems (18)–(21). □

The above variable reduction technique can also be applied to the reformulation (P2($\theta$)). The $w$ vector in problem (P2($\theta$)) can also be eliminated using the equality constraint (5), resulting in the following reformulation with a reduced number of variables,

$$\begin{aligned}
(\mathrm{P2R}(\theta))\quad \min_{x,y,z}\ & f_{2r}\triangleq f+\sum_{i=1}^N\theta_i(a_i^2y_i^2+2z_i^2+2a_iA_ixy_i-2A_ixz_i\\
&\qquad-2a_iy_iz_i-a_i^2y_i)\\
\text{s.t.}\ & (2),(3),(10),(6),(15).
\end{aligned}$$

**Example 2.** Consider the same original problem in Example 1. For given $\theta$, the corresponding reformulation (P2R($\theta$)) is:

$$\begin{aligned}
\min_{x,y,z}\ & x^T\begin{bmatrix}7 & 3\\ 3 & 6\end{bmatrix}x+\begin{bmatrix}-7\\ -5\end{bmatrix}^Tx+40y\\
&+\theta(5^2y^2+2z^2+2\cdot 5(-8x_1+x_2)y-2(-8x_1+x_2)z\\
&-2\cdot 5yz-5^2y)\\
\text{s.t.}\ & 5\le -8x_1+x_2-z+5y,\\
&-5x_1+2x_2-6y\le 3,\\
&(5--30)y\le z\le 5y,\\
&y\in\{0,1\},x=(x_1,x_2)^T\in\mathbb{R}^2,z\in\mathbb{R}.
\end{aligned}$$

The best parameters $\theta$ for (P2R($\theta$)) can be found by solving the following problem:

$$\max_{\theta\in\mathbb{R}^m}\{\mathrm{v}(\overline{\mathrm{P2R}}(\theta))\mid f_{2r}\text{ is a convex function.}\} \tag{24}$$

Note that $f_{1r}$ is convex if and only if the following condition on its Hessian holds:

$$Q_{2r}\triangleq\begin{pmatrix} Q & A^T\mathrm{Diag}(a)\mathrm{Diag}(\theta) & -A^T\mathrm{Diag}(\theta)\\ \mathrm{Diag}(\theta)\mathrm{Diag}(a)A & \mathrm{Diag}(a)\mathrm{Diag}(\theta)\mathrm{Diag}(a) & -\mathrm{Diag}(a)\mathrm{Diag}(\theta)\\ -\mathrm{Diag}(\theta)A & -\mathrm{Diag}(a)\mathrm{Diag}(\theta) & 2\mathrm{Diag}(\theta)\end{pmatrix}\succeq 0. \tag{25}$$

The tightness of the reduced reformulation (P2R($\theta$)) is guaranteed by the following proposition.

**Proposition 3.** v*(24)* ≥ v*(13)*.

**Proof.** The proof is similar to that of Proposition 1. □

**Proposition 4.** *Problem (24) is equivalent to the following SDP problem:*

$$\max_{\eta,\beta,\mu,\sigma,\lambda,\pi,\theta,\tau}\ \tau \tag{26}$$

$$\text{s.t.}\quad\begin{pmatrix} Q_{2r} & 0.5G\\ 0.5G^T & -\eta^Td+\beta^Ta-e^T\pi-\tau\end{pmatrix}\succeq 0,$$

$$\eta\in\mathbb{R}^M_+,\ \beta\in\mathbb{R}^m_+,\ \mu\in\mathbb{R}^m_+,\ \sigma\in\mathbb{R}^m_+,\ \lambda\in\mathbb{R}^m_+,\ \pi\in\mathbb{R}^m_+, \tag{27}$$

$$\theta\in\mathbb{R}^m,\ \tau\in\mathbb{R}, \tag{28}$$

*where*

$$G = \begin{pmatrix} c + B^T\eta - A^T\beta \\ h - \mathrm{Diag}(a)\mathrm{Diag}(a)\theta + D^T\eta - \mathrm{Diag}(a)\beta + \mathrm{Diag}(a-b)\mu \\ \quad -\mathrm{Diag}(a)\sigma + \pi - \lambda \\ \beta - \mu + \sigma \end{pmatrix}.$$

**Proof.** Associate the following multipliers to the constraints in $(\overline{\mathrm{P2R}}(\theta))$:

- $\eta \in \mathbb{R}_+^M$ for $Bx + Dy \le d$;
- $\beta_i \in \mathbb{R}_+$ for $a_i \le A_i x - z_i + a_i y_i,\ i = 1,\dots,m$;
- $\mu_i$ and $\sigma_i \in \mathbb{R}_+$ for $(a_i - b_i)y_i \le z_i$ and $z_i \le a_i y_i$, respectively, $i = 1,\dots,m$;
- $\lambda_i$ and $\pi_i \in \mathbb{R}_+$ for $0 \le y_i$ and $y_i \le 1$, $i = 1,\dots,m$;

Let $\mu = (\mu_1,\dots,\mu_m)^T$, $\sigma = (\sigma_1,\dots,\sigma_m)^T$, $\beta = (\beta_1,\dots,\beta_m)^T$, $\lambda = (\lambda_1,\dots,\lambda_m)^T$ and $\pi = (\pi_1,\dots,\pi_m)^T$. The Lagrangian function of $(\overline{\mathrm{P2R}}(\theta))$ is then given by

$$\begin{aligned} L(\cdot) &= L(x,y,z;\eta,\beta,\mu,\sigma,\pi,\lambda;\theta) \\ &= f_{2r} + \eta^T(Bx + Dy - d) + \sum_{i=1}^m \beta_i(a_i - a_i y_i - A_i x + z_i) \\ &\quad + \sum_{i=1}^m \mu_i((a_i - b_i)y_i - z_i) + \sum_{i=1}^m \sigma_i(z_i - a_i y_i) \\ &\quad + \sum_{i=1}^m \pi_i(y_i - 1) - \sum_{i=1}^m \lambda_i y_i \\ &= (x^T, y^T, z^T)Q_{1r}(x^T, y^T, z^T)^T \\ &\quad + (c + B^T\eta - A^T\beta)^T x \\ &\quad + (h - \mathrm{Diag}(a)\mathrm{Diag}(a)\theta + D^T\eta - \mathrm{Diag}(a)\beta + \mathrm{Diag}(a-b)\mu \\ &\quad - \mathrm{Diag}(a)\sigma + \pi - \lambda)^T y + (\beta - \mu + \sigma)^T z \\ &\quad - \eta^T d + \beta^T a - e^T\pi. \end{aligned}$$

Because the Slater condition is automatically satisfied due to the linearity in constraints, problem (24) is equivalent to the following problem by Lagrangian duality:

$$\max_{\theta\in\mathbb{R}^m,(25),(27)} \left\{ \min_{(x,y,z)\in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m} L(\cdot) \right\}. \tag{29}$$

If (25) does not hold, then the inner minimization of problem (29) is an unconstrained non-convex quadratic minimization problem and its optimal value is $-\infty$. Thus, problem (29) can be simplified to:

$$\max_{\theta\in\mathbb{R}^m,(27)} \left\{ \min_{(x,y,z)\in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m} L(\cdot) \right\}. \tag{30}$$

Following Corollary 4.2 in Lemaréchal and Oustry (1999), problem (30) is equivalent to the SDP problems (26)–(28). □

## 4. New quadratic convex reformulation

In this section, we extend the QCR approach in the literature to derive a better equivalent reformulation for problem (P). The applicability of the QCR approach in the literature is limited to problems with equality constraints (The binary constraint $y_i \in \{0, 1\}$ is equivalent to $y_i^2 - y_i = 0$). This means that we cannot apply the QCR approach directly to problem (P), since merely adding $y_i^2 - y_i$ to the objective function would not help improve the lower bound if the convexity of the objective function is to be maintained. We need to generalize the QCR approach so that it can be applied to problem (P). Consider the following reformulation:

$$\begin{aligned} (\mathrm{P}(q)) \quad \min_{x,y,z} \quad & f_q \triangleq f + \sum_{i=1}^m q_i(A_i x, z_i, y_i) \\ \text{s.t.} \quad & A_i x - a_i \le z_i \le A_i x - a_i + b_i y_i,\ i = 1,\dots,m, && (31) \\ & 0 \le z, && (32) \\ & (2), (3), (10), \end{aligned}$$

where $q_i(\cdot,\ \cdot,\ \cdot)$ is a quadratic function on three variables. Note that the original objective function of (P) only has quadratic terms on $x$. Our goal is to lift the objective function so that the quadratic terms of all decision variables can be utilized to improve the reformulation. That is why we will add a quadratic function $q_i(\cdot,\ \cdot,\ \cdot)$ that has quadratic terms of both $y$ and $z$.

The first question is what quadratic functions we can add to the objective. This is not a trivial question as we need to maintain the equivalence to the original problem (P). We start from the following proposition.

**Proposition 5.** *If $q_i(A_i x, A_i x - a_i, 0) = 0$ and $q_i(A_i x, z_i, 1)$ is a positive multiple of $z_i(z_i - (A_i x - a_i))$, i.e., $q_i(A_i x, z_i, 1) = K_i z_i(z_i - (A_i x - a_i))$ for some constant $K_i > 0$, then the following holds true.*

*(a) If $(x,\ y)$ is a feasible solution for (P), let $z_i = \max\{A_i x - a_i, 0\}$, $i = 1,\dots,m$, then $(x,\ y,\ z)$ is a feasible solution for $(\mathrm{P}(q))$ and $f(x) = f_q(x, y, z)$;*

*(b) If $(x^*,\ y^*,\ z^*)$ is an optimal solution for $(\mathrm{P}(q))$, then $(x^*,\ y^*)$ is a feasible solution for (P) and $f(x^*) = f_q(x^*, y^*, z^*)$;*

*(c) $\mathrm{v}(\mathrm{P}(q)) = \mathrm{v}(\mathrm{P})$.*

**Proof.** (a) If $y_i = 0$, then from constraint (1), we have $A_i x \ge a_i$. Now the constraint (31) will become

$$A_i x - a_i \le z_i \le A_i x - a_i.$$

Thus $z_i = \max\{A_i x - a_i, 0\} = A_i x - a_i$ satisfies the constraints (31) and (32).

If $y_i = 1$, then constraint (1) becomes $A_i x - a_i + b_i \ge 0$. Now the constraint (31) becomes

$$A_i x - a_i \le z_i \le A_i x - a_i + b_i.$$

Because $A_i x - a_i + b_i \ge 0$, $z_i = \max\{A_i x - a_i, 0\}$ also satisfies the constraints (31) and (32). Hence $(x,\ y,\ z)$ is feasible for $(\mathrm{P}(q))$. Because $q_i(A_i x,\ z_i,\ 1)$ is a positive multiple of $z_i(z_i - (A_i x - a_i))$, we have $q_i(A_i x, z_i, 1) = 0$ and thus $f(x) = f_q(x, y, z)$.

(b) Since $(x^*,\ y^*,\ z^*)$ is feasible for $(\mathrm{P}(q))$, the constraints (31) and (32) ensure that $0 \le z_i^* \le A_i x^* - a_i - b_i y_i^*$, $i = 1,\dots,m$. Thus $(x^*,\ y^*)$ satisfies constraint (1) and is feasible for (P). Next, since $(x^*,\ y^*,\ z^*)$ is optimal for $(\mathrm{P}(q))$, $z^*$ must be optimal for the following problem:

$$\min_z \quad \sum_{i=1}^m q_i(A_i x^*, z_i, y_i^*) \tag{33}$$

$$\text{s.t.} \quad A_i x^* - a_i \le z_i \le A_i x^* - a_i + b_i y_i^*,\ i = 1,\dots,m, \tag{34}$$

$$0 \le z_i,\ i = 1,\dots,m, \tag{35}$$

$$(10).$$

From the assumption we know that $q_i(A_i x^*, A_i x^* - a_i, 0) = 0$ and $q_i(A_i x^*,\ z_i,\ 1)$ is a positive multiple of $z_i(z_i - (A_i x^* - a_i))$ for all $i = 1,\dots,m$. Because we are minimizing the objective function, if $y_i^* = 0$, then $z_i^* = A_i x^* - a_i$; if $y_i^* = 1$, then $z_i^* = \max\{A_i x^* - a_i, 0\}$. Thus v((33)–(35))=0 and $f(x^*) = f_q(x^*, y^*, z^*)$.

(c) Let $(\bar{x}, \bar{y})$ be the optimal solution for (P) and $(x^*,\ y^*,\ z^*)$ be an optimal solution for $(\mathrm{P}(q))$. Define $\bar{z}_i = \max\{A_i\bar{x} - a_i, 0\}$, $i = 1,\dots,m$. Then from the proof for part (a), $(\bar{x}, \bar{y}, \bar{z})$ is also feasible for $(\mathrm{P}(q))$ and $f(\bar{x}) = f_q(\bar{x}, \bar{y}, \bar{z})$. Thus we have $\mathrm{v}(\mathrm{P}) = f(\bar{x}) =$

$f_q(\bar{x},\bar{y},\bar{z}) \geq f_q(x^*,y^*,z^*) = \mathrm{v}(\mathrm{P}(q))$. On the other hand, from the proof for part (b), $(x^*, y^*)$ is also feasible for (P) and $f(x^*) = f_q(x^*,y^*,z^*)$. Thus we have $\mathrm{v}(\mathrm{P}) = f(\bar{x}) \leq f(x^*) = f_q(x^*,y^*,z^*) = \mathrm{v}(\mathrm{P}(q))$. □

**Corollary 1.** *If $q_i(A_ix, A_ix-a_i, 0) = 0$ and $q_i(A_ix, z_i, 1)$ is a positive multiple of $z_i(z_i-(A_ix-a_i))$, then the following holds true.*

*(a) If $(x^*, y^*)$ is an optimal solution for (P), let $z_i = \max\{A_ix^*-a_i, 0\}$ then $(x^*, y^*, z)$ is an optimal solution for (P(q));*

*(b) If $(x^*, y^*, z^*)$ is an optimal solution for (P(q)), then $(x^*, y^*)$ is an optimal solution for (P).*

From the proof of Proposition 5, we can find that one way to maintain the equivalence between problems (P(q)) and (P) is to require that: (a) the quadratic function $q_i(\cdot,\cdot,\cdot)$ vanishes on the set of optimal solutions for (P(q)); (b) the projection of the optimal solution set for (P(q)) onto the (x, y) space is a subset of the optimal solution set for (P). We do not need to require the quadratic function $q_i(\cdot,\cdot,\cdot)$ to vanish on the entire feasible region. This is a prominent feature when compared to the QCR approach in the literature.

Next we are going to find out the exact expression for $q_i(\cdot,\cdot,\cdot)$ under the conditions of Proposition 5. Assume that

$$q_i(A_ix, z_i, y_i) = k_1(A_ix)^2 + k_2z_i^2 + k_3y_i^2 + k_4A_ixz_i + k_5A_ixy_i + k_6z_iy_i + k_7A_ix + k_8z_i + k_9y_i + k_{10},$$

where $k_1, k_2, \ldots, k_{10}$ are the coefficients in the quadratic function $q_i(\cdot,\cdot,\cdot)$. Then to ensure the conditions in Theorem 5, that is, $q_i(A_ix, A_ix-a_i, 0) = 0$ and $q_i(A_ix, z_i, 1)$ is a positive multiple of $z_i(z_i-(A_ix-a_i))$, we must have

$$\begin{aligned} &q_i(A_ix, A_ix-a_i, 0) \qquad (36)\\ &= k_1(A_ix)^2 + k_2(A_ix-a_i)^2 + 0 + k_4A_ix(A_ix-a_i) + 0 + 0\\ &\quad + k_7A_ix + k_8(A_ix-a_i) + 0 + k_{10},\\ &= (A_ix)^2(k_1+k_2+k_4) + (A_ix)(-2a_ik_2 - a_ik_4 + k_7 + k_8)\\ &\quad + a_i^2k_2 - a_ik_8 + k_{10}\\ &= 0, \end{aligned} \tag{36}$$

and

$$\begin{aligned} &q_i(A_ix, z_i, 1) \qquad (37)\\ &= k_1(A_ix)^2 + k_2z_i^2 + k_3 + k_4A_ixz_i + k_5A_ix + k_6z_i\\ &\quad + k_7A_ix + k_8z_i + k_9 + k_{10},\\ &= (A_ix)^2(k_1) + z_i^2(k_2) + A_ixz_i(k_4) + A_ix(k_5+k_7) + z_i(k_6+k_8)\\ &\quad + k_3 + k_9 + k_{10}\\ &= Kz_i(z_i-(A_ix-a_i)). \end{aligned}$$

If the equalities (36) and (37) hold for any $A_i$ and $x$, then the coefficients must satisfy the following constraints:

$$\begin{aligned} k_1+k_2+k_4 &= 0, & k_1 &= 0,\\ -2a_ik_2 - a_ik_4 + k_7 + k_8 &= 0, & k_5+k_7 &= 0,\\ a_i^2k_2 - a_ik_8 + k_{10} &= 0, & k_3+k_9+k_{10} &= 0,\\ && -K &= k_4,\\ && k_6+k_8 &= Ka_i,\\ && k_2 &= K,\\ && K &\geq 0, \end{aligned}$$

which can be simplified to:

$$\begin{aligned} k_1 &= 0, & k_8 &= a_ik_2 + k_5,\\ k_4 &= -k_2, & k_9 &= -k_3 - a_ik_5,\\ k_6 &= -k_5, & k_{10} &= a_ik_5,\\ k_7 &= -k_5, & k_2 &\geq 0. \end{aligned}$$

As all the coefficients will be fixed once $k_2$, $k_3$ and $k_5$ are fixed, the degree of freedom of the coefficients is three. Replacing the letters $k_2$, $k_3$, $k_6$ by $u_i, v_i, w_i$, we have the following form for $q_i(A_ix, z_i, y_i)$:

$$\begin{aligned} q_i(A_ix, z_i, y_i; u_i, v_i, w_i) = {} & v_iy_i^2 + u_iz_i^2 + w_iA_ixy_i - u_iA_ixz_i\\ & - w_iy_iz_i - w_iA_ix + (a_iu_i + w_i)z_i\\ & - (v_i + a_iw_i)y_i + a_iw_i, \end{aligned} \tag{38}$$

where $u_i, v_i$ and $w_i$ are the three coefficients for $q_i(\cdot,\cdot,\cdot)$ and $u_i \geq 0$.

From now on we assume that $q_i(\cdot,\cdot,\cdot)$ takes the form in (38). Then by Proposition 5, we have a set of equivalent reformulations of (P) parameterized by $(u, v, w)$, where $u = (u_1, \ldots, u_m)^T, v = (v_1, \ldots, v_m)^T$ and $w = (w_1, \ldots, w_m)^T$. Denote the corresponding reformulation by $(\mathrm{P}_q(u, v, w))$.

**Example 3.** Consider the same original problem in Example 1. For given $(u, v, w)$, the corresponding reformulation $(\mathrm{P}_q(u, v, w)))$ is:

$$\begin{aligned} \min_{x,y,z}\ & x^T\begin{bmatrix}7 & 3\\ 3 & 6\end{bmatrix}x + \begin{bmatrix}-7\\ -5\end{bmatrix}^T x + 40y\\ & + vy^2 + uz^2 + w(-8x_1+x_2)y - u(-8x_1+x_2)z\\ & - wyz - w(-8x_1+x_2)\\ & + (5u+w)z - (v+5w)y + 5w\\ \text{s.t.}\ & -8x_1 + x_2 - 5 \leq z \leq -8x_1 + x_2 - 5 + 30y,\\ & 0 \leq z,\\ & y \in \{0,1\}, x = (x_1,x_2)^T \in \mathbb{R}^2, z \in \mathbb{R}. \end{aligned}$$

We are interested in finding the best[4] parameters $(u, v, w)$ by solving the following problem:

$$\max_{u,v,w}\ \mathrm{v}(\overline{\mathrm{P}}_q(u, v, w)) \tag{39}$$

$$\text{s.t.}\quad u \in \mathbb{R}_+^m, v \in \mathbb{R}^m, w \in \mathbb{R}^m, \tag{40}$$

$$f_q \text{ is a convex function.} \tag{41}$$

Note that $f_q$ is convex if and only if the following condition on its Hessian holds:

$$Q_q \triangleq \begin{pmatrix} Q & \frac{1}{2}A^T\mathrm{Diag}(w) & -\frac{1}{2}A^T\mathrm{Diag}(u)\\ \frac{1}{2}\mathrm{Diag}(w)A & \mathrm{Diag}(v) & -\frac{1}{2}\mathrm{Diag}(w)\\ -\frac{1}{2}\mathrm{Diag}(u)A & -\frac{1}{2}\mathrm{Diag}(w) & \mathrm{Diag}(u) \end{pmatrix} \succeq 0. \tag{42}$$

**Proposition 6.** *Problems (39)–(41) is equivalent to the following SDP problem:*

$$\max_{\eta,\mu,\sigma,\lambda,\pi,\rho,u,v,w,\tau}\ \tau \tag{43}$$

$$\begin{aligned} \text{s.t.}\quad & \begin{pmatrix} Q_q & 0.5G\\ 0.5G^T & a^Tw - d^T\eta - a^T\mu + a^T\sigma - e^T\pi - \tau \end{pmatrix} \succeq 0,\\ & (\eta,\mu,\sigma,\lambda,\pi,\rho) \in \mathbb{R}_+^M \times \mathbb{R}_+^m \times \mathbb{R}_+^m \times \mathbb{R}_+^m \times \mathbb{R}_+^m \times \mathbb{R}_+^m, \end{aligned} \tag{44}$$

$$(u, v, w, \tau) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}, \tag{45}$$

*where*

$$G = \begin{pmatrix} c - A^Tw + B^T\eta + A^T\mu - A^T\sigma\\ h - v - \mathrm{Diag}(a)w + D^T\eta - \mathrm{Diag}(b)\sigma - \lambda + \pi\\ \mathrm{Diag}(a)u + w - \mu + \sigma - \rho \end{pmatrix}.$$

**Proof.** Associate the following multipliers to the constraints in $(\overline{\mathrm{P}}_q(u, v, w))$:

[4] By "best", we mean that the continuously relaxed problem should have a minimum that is as high as possible.

- $\eta \in \mathbb{R}^M_+$ for $Bx + Dy \le d$;
- $\mu_i$ and $\sigma_i \in \mathbb{R}_+$ for $A_i x - a_i \le z_i$ and $z_i \le A_i x - a_i + b_i y_i$, respectively, $i = 1, \ldots, m$;
- $\lambda_i$ and $\pi_i \in \mathbb{R}_+$ for $y_i \ge 0$ and $y_i \le 1$, respectively, $i = 1, \ldots, m$;
- $\rho_i \in \mathbb{R}_+$ for $z_i \ge 0$, $i = 1, \ldots, m$.

Let $\mu = (\mu_1, \ldots, \mu_m)^T$, $\sigma = (\sigma_1, \ldots, \sigma_m)^T$, $\lambda = (\lambda_1, \ldots, \lambda_m)^T$, $\pi = (\pi_1, \ldots, \pi_m)^T$ and $\rho = (\rho_1, \ldots, \rho_m)^T$. Then, the Lagrangian function of $(\overline{\mathrm{P}}_q(u, v, w))$ is given by

$$\begin{aligned}
L(\cdot) &= L(x, y, z; \eta, \mu, \sigma, \lambda, \pi, \rho; u, v, w)\\
&= x^T Q x + c^T x + h^T y + \sum_{i=1}^m [u_i z_i^2 + v_i y_i^2 - u_i A_i x z_i + w_i A_i x y_i\\
&\quad - w_i z_i y_i - w_i A_i x + (a_i u_i + w_i) z_i - (v_i + a_i w_i) y_i + a_i w_i]\\
&\quad + \eta^T (Bx + Dy - d) + \sum_{i=1}^m \mu_i (A_i x - a_i - z_i)\\
&\quad + \sum_{i=1}^m \sigma_i (z_i - A_i x + a_i - b_i y_i)\\
&\quad + \sum_{i=1}^m \lambda_i (-y_i) + \sum_{i=1}^m \pi_i (y_i - 1) - \rho^T z\\
&= x^T Q x + (c - A^T w + B^T \eta + A^T \mu - A^T \sigma)^T x\\
&\quad + y^T \mathrm{Diag}(v) y + (h - v - \mathrm{Diag}(a) w + D^T \eta\\
&\quad - \mathrm{Diag}(b) \sigma - \lambda + \pi)^T y\\
&\quad + z^T \mathrm{Diag}(u) z + (\mathrm{Diag}(a) u + w - \mu + \sigma - \rho)^T z\\
&\quad + x^T (A^T \mathrm{Diag}(w)) y + x^T (-A^T \mathrm{Diag}(u)) z + z^T (-\mathrm{Diag}(w)) y\\
&\quad + a^T w - d^T \eta - a^T \mu + a^T \sigma - e^T \pi.
\end{aligned}$$

Because the Slater condition is satisfied, problems (39)–(41) is equivalent to the following problem by Lagrangian duality:

$$\max_{(u,v,w) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m, (42), (44)} \left\{ \min_{(x,y,z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m} L(\cdot) \right\}. \tag{46}$$

If (42) does not hold, then the inner minimization of problem (46) is an unconstrained non-convex quadratic minimization problem and its optimal value is $-\infty$. Thus, problem (46) can be simplified to:

$$\max_{(u,v,w) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m, (44)} \left\{ \min_{(x,y,z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m} L(\cdot) \right\}. \tag{47}$$

Following Corollary 4.2 in Lemaréchal and Oustry (1999), problem (47) is equivalent to the SDP problems (43)–(45). □

## 5. Tightness of the new reformulation

In order to explore the tightness of our new reformulation, we compare in this section our new reformulation with (P2R($\theta$)) and then (P1R($\theta$)), both of which are presented in Section 2.

In fact, (P2R($\theta$)) is a special case of the new reformulation (P($q$)). To see this, we replace the variable $z_i$ in (P($q$)) by $A_i x - a_i - z_i + a_i y_i$ and get the following equivalent reformulation:

$$\begin{aligned}
(\mathrm{P}'_q(u, v, w))) \quad \min_{x,y,z}\ & f'_q \triangleq f + \sum_{i=1}^m [(a_i^2 u_i + v_i - a_i w_i) y_i^2 + u_i z_i^2\\
& + a_i u_i A_i x y_i - u_i A_i x z_i + (w_i - 2a_i u_i) y_i z_i\\
& + (-a_i^2 u_i + a_i w_i - v_i) y_i) + (a_i u_i - w_i) z_i]\\
\text{s.t.}\ & (2), (3), (10), (6), (15).
\end{aligned}$$

Then one can check that (P2R($\theta$)) is a special case of the above problem with the following simple substitutions,

$$\begin{aligned}
u_i &= 2\theta_i\\
v_i &= a_i^2 \theta_i\\
w_i &= 2a_i \theta_i.
\end{aligned}$$

Thus the objective value of problem (24) would be smaller or equal to the objective value of the SDP problems (43)–(45) and our new reformulation will be tighter than (P2R($\theta$)). That is, the lower bound from the continuous relaxation of (P2R($\theta$)) where $\theta$ is from problem (24) is not as tight as the lower bound from the continuous relaxation of our new reformulation ($\mathrm{P}_q(u^*, v^*, w^*)$) where $(u^*, v^*, w^*)$ is extracted from the optimal solution of the SDP problems (43)–(45).

Next, we compare our new reformulation with (P1R($\theta$)).

**Proposition 7.** *Suppose that $\theta^*$ is an optimal solution for problem (16), and $(x^*, y^*, z^*, \phi^*)$ is an optimal solution for $(\overline{\mathrm{P1R}}(\theta^*))$. Define $\overline{u}, \overline{v}, \overline{w} \in \mathbb{R}^m$ with*

$$\overline{u}_i = 2\theta_i^*, \ \ i = 1, \ldots, m, \tag{48}$$

$$\overline{v}_i = \begin{cases} \theta_i^* a_i^2 & \text{if } y_i^* = 0,\\ \theta_i^* (a_i - \frac{z_i^*}{y_i^*})^2 & \text{otherwise,} \end{cases} \quad i = 1, \ldots, m, \tag{49}$$

$$\overline{w}_i = \begin{cases} 2\theta_i^* a_i & \text{if } y_i^* = 0,\\ 2\theta_i^* (a_i - \frac{z_i^*}{y_i^*}) & \text{otherwise,} \end{cases} \quad i = 1, \ldots, m. \tag{50}$$

*Then the following holds true.*

*(a) The objective function of $(\mathrm{P}'_q(\overline{u}, \overline{v}, \overline{w}))$is convex;*
*(b) $(x^*, y^*, z^*)$ is also optimal for $(\overline{\mathrm{P}'}_q(\overline{u}, \overline{v}, \overline{w}))$;*
*(c) $\mathrm{v}(\mathrm{P}'_q(\overline{u}, \overline{v}, \overline{w})) = \mathrm{v}(\overline{\mathrm{P1R}}(\theta^*))$.*

**Proof.** a) For the convexity of the objective function, we only need to consider the quadratic terms. Because $\theta^*$ is a feasible solution for problem (16), the objective function of $(\overline{\mathrm{P1R}}(\theta^*))$ must be convex. Compared with the quadratic terms of the objective function of $(\overline{\mathrm{P1R}}(\theta^*))$, the objective function of $(\mathrm{P}'_q(\overline{u}, \overline{v}, \overline{w}))$ has additional quadratic terms

$$\sum_{i=1}^m \theta_i^* (\frac{z_i^*}{y_i^*} y_i - z_i)^2.$$

Since $\theta^* \ge 0$, the objective function of $(\mathrm{P}'_q(\overline{u}, \overline{v}, \overline{w}))$ is a summation of convex functions and thus must be convex.

b) From the formulation of problem $(\overline{\mathrm{P1R}}(\theta^*))$, we must have

$$\phi_i^* = \frac{z_i^{*2}}{y_i^*}, \tag{51}$$

since $\theta^* \ge 0$. Consider the following auxiliary problem:

$$\begin{aligned}
(\mathrm{P1R}'(\theta^*)) \ \ \min_{x,y,z}\ & f'_{1r} \triangleq f + \sum_{i=1}^m \theta_i^* (a_i^2 y_i^2 + z_i^2 + 2a_i A_i x y_i - 2A_i x z_i\\
& - 2a_i y_i z_i + \frac{z_i^2}{y_i} - a_i^2 y_i)\\
\text{s.t.}\ & (2), (3), (10), (6), (15).
\end{aligned}$$

As $(x^*, y^*, z^*, \phi^*)$ is an optimal solution of $(\overline{\mathrm{P1R}}(\theta^*))$, $(x^*, y^*, z^*)$ is also feasible for $(\overline{\mathrm{P1R}'}(\theta^*))$ and $f_{1r}(x^*, y^*, z^*, \phi^*) = f'_{1r}(x^*, y^*, z^*)$ due to Eq. (51), thus

$$\mathrm{v}(\overline{\mathrm{P1R}}(\theta^*)) = f_{1r}(x^*, y^*, z^*, \phi^*) = f'_{1r}(x^*, y^*, z^*) \ge \mathrm{v}(\overline{\mathrm{P1R}'}(\theta^*)). \tag{52}$$

On the other hand, if $(\hat{x}, \hat{y}, \hat{z})$ is optimal for $(\overline{\text{P1R}'}(\theta^*))$, then $(\hat{x}, \hat{y}, \hat{z}, \hat{\phi})$ is feasible for $(\overline{\text{P1R}}(\theta^*))$, where

$$\hat{\phi}_i \triangleq \begin{cases} 0 & \text{if } \hat{y}_i = 0, \\ \frac{\hat{z}_i^2}{\hat{y}_i} & \text{if } \hat{y}_i > 0. \end{cases} \quad i = 1, \ldots, m. \tag{53}$$

Also, by comparing the two objective functions, we have $f_{1r}(\hat{x}, \hat{y}, \hat{z}, \hat{\phi}) = f'_{1r}(\hat{x}, \hat{y}, \hat{z})$. Hence

$$\text{v}(\overline{\text{P1R}}(\theta^*)) \le f_{1r}(\hat{x}, \hat{y}, \hat{z}, \hat{\phi}) = f'_{1r}(\hat{x}, \hat{y}, \hat{z}) = \text{v}(\overline{\text{P1R}'}(\theta^*)). \tag{54}$$

From inequalities (52) and (54), we have $\text{v}(\overline{\text{P1R}'}(\theta^*)) = \text{v}(\overline{\text{P1R}}(\theta^*))$ and thus $(x^*, y^*, z^*)$ is also optimal for $(\overline{\text{P1R}'}(\theta^*))$.

To show that $(x^*, y^*, z^*)$ is also optimal for $(\overline{\text{P}'}_q(\overline{u}, \overline{v}, \overline{w}))$, we compare the gradients of the objective functions of $(\overline{\text{P}'}_q(\overline{u}, \overline{v}, \overline{w}))$ and $(\overline{\text{P1R}'}(\theta^*))$ at the point $(x^*, y^*, z^*)$. Denote the component in a gradient vector $\nabla g$ of a function $g$ corresponding to the variable $x_i$ by $(\nabla g)_{x_i}$. Then we have

$$\begin{aligned}
&(\nabla f'_q(x^*, y^*, z^*; \overline{u}, \overline{v}, \overline{w}))_{x_i} \\
&= \frac{\partial (x^T Q x + c^T x + \sum_{i=1}^m (a_i \overline{u}_i A_i x y_i - \overline{u}_i A_i x z_i))}{\partial x_i}\Big|_{x^*, y^*, z^*} \\
&= \frac{\partial (x^T Q x + c^T x + \sum_{i=1}^m \theta_i^* (2 a_i A_i x y_i - 2 A_i x z_i))}{\partial x_i}\Big|_{x^*, y^*, z^*} \\
&= (\nabla f'_{1r}(x^*, y^*, z^*; \theta^*))_{x_i},
\end{aligned}$$

$$\begin{aligned}
&(\nabla f'_q(x^*, y^*, z^*; \overline{u}, \overline{v}, \overline{w}))_{y_i} \\
&= \frac{\begin{array}{l}\partial (\sum_{i=1}^m ((a_i^2 \overline{u}_i + \overline{v}_i - a_i \overline{w}_i) y_i^2 + a_i \overline{u}_i A_i x y_i \\ \quad + (\overline{w}_i - 2 a_i \overline{u}_i) y_i z_i + (-a_i^2 \overline{u}_i + a_i \overline{w}_i - \overline{v}_i) y_i))\end{array}}{\partial y_i}\Big|_{x^*, y^*, z^*} \\
&= \sum_{i=1}^m \left( 2\theta_i^* \left( 2a_i^2 + \left( a_i - \frac{z_i^*}{y_i^*} \right)^2 - a_i 2 \left( a_i - \frac{z_i^*}{y_i^*} \right) \right) y_i + a_i 2 \theta_i^* A_i x \right. \\
&\quad + 2\theta_i^* \left( \left( a_i - \frac{z_i^*}{y_i^*} \right) - 2a_i \right) z_i + \theta_i^* \left( -2a_i^2 + a_i 2 \left( a_i - \frac{z_i^*}{y_i^*} \right) \right. \\
&\quad \left.\left. - \left( a_i - \frac{z_i^*}{y_i^*} \right)^2 \right) \right) \Big|_{x^*, y^*, z^*}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^m \left( 2\theta_i^* \left( a_i^2 + \frac{z_i^{*2}}{y_i^{*2}} \right) y_i^* + a_i 2 \theta_i^* A_i x^* \right. \\
&\quad \left. + 2\theta_i^* \left( -\frac{z_i^*}{y_i^*} - a_i \right) z_i^* - \theta_i^* \left( a_i^2 + \frac{z_i^{*2}}{y_i^{*2}} \right) \right) \\
&= \sum_{i=1}^m \theta_i^* \left( 2a_i^2 y_i^* + 2a_i A_i x^* - 2a_i z_i^* - \frac{z_i^{*2}}{y_i^{*2}} - a_i^2 \right) \\
&= \sum_{i=1}^m \theta_i^* \left( 2a_i^2 y_i + 2a_i A_i x - 2a_i z_i - \frac{z_i^2}{y_i^2} - a_i^2 \right) \Big|_{x^*, y^*, z^*} \\
&= \frac{\partial \left( \sum_{i=1}^N \theta_i^* \left( a_i^2 y_i^2 + 2a_i A_i x y_i - 2a_i y_i z_i + \frac{z_i^2}{y_i} - a_i^2 y_i \right) \right)}{\partial y_i}\Big|_{x^*, y^*, z^*} \\
&= (\nabla f'_{1r}(x^*, y^*, z^*; \theta^*))_{y_i},
\end{aligned}$$

and

$$\begin{aligned}
&(\nabla f'_q(x^*, y^*, z^*; \overline{u}, \overline{v}, \overline{w}))_{z_i} \\
&= \frac{\partial \left( \sum_{i=1}^m (\overline{u}_i z_i^2 - \overline{u}_i A_i x z_i + (\overline{w}_i - 2a_i \overline{u}_i) y_i z_i + (a_i \overline{u}_i - \overline{w}_i) z_i) \right)}{\partial z_i}\Big|_{x^*, y^*, z^*} \\
&= \sum_{i=1}^N \left( 4\theta_i^* z_i - 2\theta_i^* A_i x + 2\theta_i^* \left( \left( a_i - \frac{z_i^*}{y_i^*} \right) - 2a_i \right) y_i \right. \\
&\quad \left. + \left( 2a_i \theta_i^* - 2\theta_i^* \left( a_i - \frac{z_i^*}{y_i^*} \right) \right) \right) \Big|_{x^*, y^*, z^*} \\
&= \sum_{i=1}^m \left( 4\theta_i^* z_i - 2\theta_i^* A_i x^* + 2\theta_i^* \left( -a_i - \frac{z_i^*}{y_i^*} \right) y_i^* \right. \\
&\quad \left. + \left( 2a_i \theta_i^* - 2\theta_i^* \left( a_i - \frac{z_i^*}{y_i^*} \right) \right) \right) \\
&= \sum_{i=1}^m \theta_i^* \left( 2z_i^* - 2A_i x^* - 2a_i y_i^* + 2\frac{z_i^*}{y_i^*} \right) \\
&= \sum_{i=1}^m \theta_i^* \left( 2z_i - 2A_i x - 2a_i y_i + 2\frac{z_i}{y_i} \right) \Big|_{x^*, y^*, z^*} \\
&= \frac{\partial \left( \sum_{i=1}^m \theta_i^* \left( z_i^2 - 2A_i x z_i - 2a_i y_i z_i + \frac{z_i^2}{y_i} \right) \right)}{\partial z_i}\Big|_{x^*, y^*, z^*} \\
&= (\nabla f'_{1r}(x^*, y^*, z^*; \theta^*))_{z_i}.
\end{aligned}$$

So $\nabla f'_q(x^*, y^*, z^*; \overline{u}, \overline{v}, \overline{w}) = \nabla f'_{1r}(x^*, y^*, z^*; \theta^*)$. As $(x^*, y^*, z^*)$ is optimal to problem $(\overline{\text{P1R}'}(\theta^*))$, the directional derivative of the objective function at $(x^*, y^*, z^*)$ along any feasible direction should be non-negative. Since the feasible regions of $(\overline{\text{P}'}_q(\overline{u}, \overline{v}, \overline{w}))$ and $(\overline{\text{P1R}'}(\theta^*))$ are the same, the directional derivative of the objective function of $(\overline{\text{P}'}_q(\overline{u}, \overline{v}, \overline{w}))$ at $(x^*, y^*, z^*)$ along any feasible direction is also non-negative. So $(x^*, y^*, z^*)$ must also be optimal for $(\overline{\text{P}'}_q(\overline{u}, \overline{v}, \overline{w}))$ because of the convexity of the objective function. (See e.g., Chapter 2.1 of Borwein & Lewis, 2006.)

c) From (b), we only need to compare the objective values of $(\overline{\text{P1R}'}(\theta^*))$ and $(\overline{\text{P}'}_q(\overline{u}, \overline{v}, \overline{w}))$ at the optimal point $(x^*, y^*, z^*)$:

$$\begin{aligned}
&f'_{1r}(x^*, y^*, z^*; \theta^*) - f'_q(x^*, y^*, z^*; \overline{u}, \overline{v}, \overline{w}) \\
&= f + \sum_{i=1}^m \theta_i^* \left( a_i^2 y_i^{*2} + z_i^{*2} + 2a_i A_i x^* y_i^* - 2A_i x^* z_i^* - 2a_i y_i^* z_i^* \right. \\
&\quad \left. + \frac{z_i^{*2}}{y_i^*} - a_i^2 y_i^* \right) \\
&\quad - \left( f + \sum_{i=1}^m \left( \overline{u}_i z_i^{*2} + (a_i^2 \overline{u}_i + \overline{v}_i - a_i \overline{w}_i) y_i^2 + a_i \overline{u}_i A_i x^* y_i^* - \overline{u}_i A_i x^* z_i^* \right. \right. \\
&\quad \left. \left. + (\overline{w}_i - 2a_i \overline{u}_i) y_i^* z_i^* + (-a_i^2 \overline{u}_i + a_i \overline{w}_i - \overline{v}_i) y_i^* + (a_i \overline{u}_i - \overline{w}_i) z_i^* \right) \right) \\
&= \sum_{i=1}^m \theta_i^* \left( a_i^2 y_i^{*2} + z_i^{*2} + 2a_i A_i x^* y_i^* - 2A_i x^* z_i^* - 2a_i y_i^* z_i^* \right. \\
&\quad \left. + \frac{z_i^{*2}}{y_i^*} - a_i^2 y_i^* \right) \\
&\quad - \left( \sum_{i=1}^m \left( \theta_i^* \left( a_i^2 + \frac{z_i^{*2}}{y_i^{*2}} \right) y_i^{*2} + 2\theta_i^* z_i^{*2} + 2a_i \theta_i^* A_i x^* y_i^* - 2\theta_i^* A_i x^* z_i^* \right. \right. \\
&\quad \left. \left. - 2\theta_i^* \left( \frac{z_i^*}{y_i^*} + a_i \right) y_i^* z_i^* - \theta_i^* \left( a_i^2 + \frac{z_i^{*2}}{y_i^{*2}} \right) y_i^* \right. \right.
\end{aligned}$$

$$+\left(2a_i\theta_i^* - 2\theta_i^*\left(a_i - \frac{z_i^*}{y_i^*}\right)\right)z_i^*\Bigg)\Bigg)$$
$$= 0.$$

This completes the proof. □

The following corollary can be easily obtained from Proposition 7.

**Corollary 2.** v*(39)* ≥ v*(16)*.

The above corollary indicates that our new reformulation is at least as tight as the reformulation (P1R($\theta$)). That is, the lower bound from the continuous relaxation of (P1R($\theta$)), where $\theta$ is from the optimal solution of problem (16), is not as tight as the lower bound from the continuous relaxation of our new reformulation ($\mathrm{P}_q(u, v, w)$) where $(u, v, w)$ is extracted from the optimal solution of problem (39).

## 6. Computational experiment

In this section we conduct numerical tests on a set of randomly generated instances to demonstrate the effectiveness of our new reformulation solved in standard MIQP solvers.

### 6.1. Test problems

We consider two sets of test problems in our experiments.

Set A: We build 30 instances for each problem size $n \in \{100, 200, 300, 400\}$ and set $m = M = n/2$. The 30 instances for each problem size are evenly divided into three groups. For different groups, the matrix $Q$ in the objective function is generated with different diagonal dominance using the method in Frangioni and Gentile (2007) and Pardalos and Rodgers (1990). The three groups are labeled using superscript in the set $\{+, 0, -\}$, from most diagonally dominant to least diagonally dominant. $c_i$ is generated uniformly from the interval: $\left[-\frac{1}{2}\sqrt{\sum_{i,j}|Q_{ij}|},\ \ \frac{1}{2}\sqrt{\sum_{i,j}|Q_{ij}|}\right]$.$h_i$ is generated uniformly from the interval: $[0,\ \Sigma_{i,j}|Q_{ij}|]$.$A_{ij}$ is generated uniformly from the interval: $[-0.5,\ \ 0.5]$.$a_i$ is generated uniformly from the interval: $[0,\ n]$.$b_i$ is generated uniformly from the interval: $[0,\ 10n]$.$B_{ij}$ and $D_{ij}$ are generated uniformly from the interval: $[-0.5,\ \ 0.5]$.$d_i$ is generated uniformly from the interval: $[-0.5n,\ \ 0.5n]$.

Set B: The second set of test problems is generated as follows: The matrix $Q$ in the objective function is generated in the same way as in Set A. $c_i$ is generated from a normal distribution with mean 0 and standard deviation $\frac{1}{2}\sqrt{\sum_{i,j}|Q_{ij}|}$.$h_i$ is generated from a normal distribution with mean $0.5\Sigma_{i,j}|Q_{ij}|$ and standard deviation $0.5\Sigma_{i,j}|Q_{ij}|$.$A_{ij}$ is generated from a normal distribution with mean 0 and standard deviation 0.5.$a_i$ is generated from a normal distribution with mean $0.5n$ and standard deviation $0.5n$.$b_i$ is generated from a normal distribution with mean $5n$ and standard deviation $5n$.$B_{ij}$ and $D_{ij}$ are generated from a normal distribution with mean 0 and standard deviation 0.5.$d_i$ is generated from a normal distribution with mean 0 and standard deviation $0.5n$.

### 6.2. Performance of reformulations

We compare the following three approaches in the numerical tests[5]:

- (P): The original formulation;
- (P2R($\theta$)): The reduced version of the MIQP reformulation in Hsia et al. (2014) with $\theta$ obtained from solving the SDP problems (26)–(28);
- ($\mathrm{P}_q(u, v, w)$):
  - For Set A: Our new reformulation with $(u, v, w)$ obtained from solving the SDP problems (43)–(45).
  - For Set B: Our new reformulation with $(\overline{u}, \overline{v}, \overline{w})$ from Proposition 7, which can be obtained by solving the SDP problems (18)–(21) and then solving the second-order cone programming (SOCP) problem ($\overline{\mathrm{P1R}}(\theta^*)$). Because the SDP problems (18)–(21) has a smaller size than the SDP problems (43)–(45), the time spent on parameter finding would be smaller. From Proposition 7, the lower bound from ($\mathrm{P}_q(\overline{u}, \overline{v}, \overline{w})$) would be the same as that from ($\overline{\mathrm{P1R}}(\theta^*)$). We use a different approach for Set B here to demonstrate that the process of parameter findings can be of interest in its own right, and how to find good parameters more efficiently could be an interesting future research topic.

The computation for Set A is conducted on a Linux server with 48 gigabytes of RAM. All the tests are confined on one single thread (2.99 gigahertz). SDP problems are solved using `sedumi` interfaced by `CVX 1.21` (CVX Research, 2012; Grant & Boyd, 2008) on `Matlab R2012b`. The three MIQP reformulations are solved in 64-bit `IBM ILOG CPLEX Optimization Studio 12.6` (Hereinafter referred to as `CPLEX`) through its `Matlab` interface. The computation for Set B is conducted on a Linux server with 64 gigabytes of RAM. All the tests are confined on one single thread (Intel(R) Xeon(R) CPU E5-2692 v2 @ 2.20 gigahertz). SDP problems are solved using `sedumi` interfaced by `CVX 2.1` on `Matlab R2015b`. The three MIQP reformulations are solved in 64-bit `IBM ILOG CPLEX Optimization Studio 12.8`[6].

We use the default setting in `CPLEX` with CPU time limit set to 3600 seconds. So the program would terminate either when the CPU time reaches 3600 seconds or when the relative gap (between the objective values of the incumbent solution and the best lower bound) is below the default tolerance threshold $10^{-4}$ (The exact value of the relative gap when `CPLEX` terminates could range between 0 and $10^{-4}$. Rounding this number to four significant digits would make it 0 or $10^{-4}$).

Tables 1 and 2 show the numerical results for the problem instances we generated. Each line reports the average results for the 10 instances in a subset. The notations in the tables are given as follows: The column "(time$_{2r}$)" is the computing time for the SDP problems (26)–(28). The column "(time$_q$)" in Table 1 is the computing time for the SDP problems (43)–(45). The column "(time$_{1r}$)" in Table 2 is the total computing time for the SDP problems (18)–(21) and the SOCP problem ($\overline{\mathrm{P1R}}(\theta^*)$). The columns "time", "gap" and "nodes" are the computing time (in seconds), relative gap and the number of nodes explored by `CPLEX` respectively. The column "max" is the maximum computing time among the 10 instances averaged. The column "min" is the minimum computing time among the 10 instances averaged. The column "uns" is number of instances unsolved among the 10 instances averaged, i.e., `CPLEX` terminates after 3600 seconds before solving those instances to optimality[7]. The column "total" under (P2R)($\theta$) is the summation of "time" and "(time$_{2r}$)". The column "total" under (P)$_q$(u, v, w) in Table 1 is the summation of "time" and "(time$_q$)".

[5] We do not report the results for the reformulation (P1R($\theta$)) here as Hsia et al. (2014) has shown that (P1($\theta$)) is strictly dominated by (P2($\theta$)) in the numerical performance. Similar results were obtained when we compared (P1R($\theta$)) and (P2R($\theta$)).

[6] Our computing environment has changed during the review process of the paper. Thus we have different computing settings for Set A and Set B. It turns out that the numerical comparisons for Set A and Set B are similar to each other.

[7] As pointed out by one referee, setting a 3600-second time limit could skew the average downwards. Our focus here is not the exact amount of time needed, but the relative performance of different reformulations.

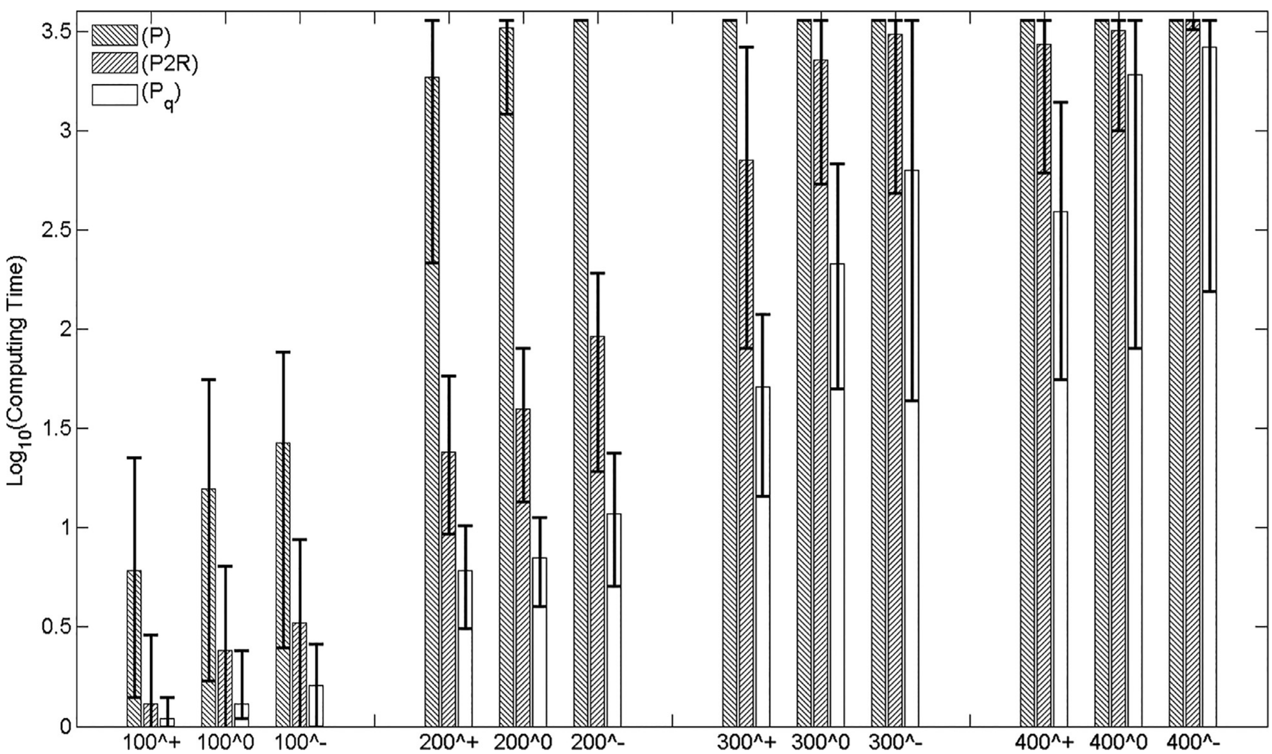


**Fig. 1.** Comparison of computing time for the test set A.

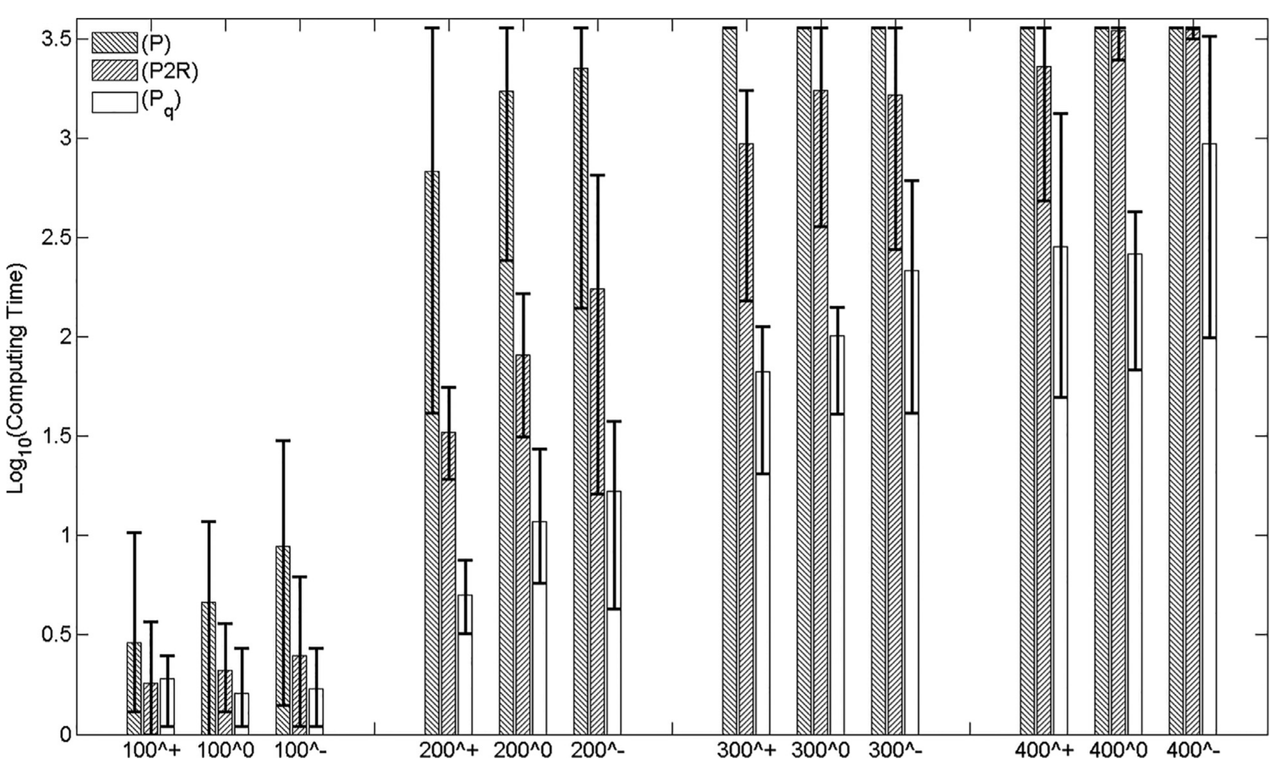


**Fig. 2.** Comparison of computing time for the test set B.

**Table 1**
Performance of MIQP reformulations on the test set A.

| $n$ | $time_{2r}$ | $time_q$ | (P) | | | | | | (P2R($\theta$)) | | | | | | | ($P_q(u,v,w)$) | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | Time | Max | Min | uns | Gap | Node | Time | Max | Min | uns | Total | Gap | Node | Time | Max | Min | uns | Total | Gap | Node |
| $100^+$ | 7.5 | 13.3 | 6.1 | 22.5 | 1.4 | 0 | 0.0000 | 3437 | 1.3 | 2.9 | 0.8 | 0 | 8.8 | 0.0000 | 85 | 1.1 | 1.4 | 1.0 | 0 | 14.4 | 0.0000 | 19 |
| $100^0$ | 7.4 | 12.7 | 15.7 | 55.7 | 1.7 | 0 | 0.0001 | 9996 | 2.4 | 6.4 | 0.9 | 0 | 9.8 | 0.0000 | 334 | 1.3 | 2.4 | 1.1 | 0 | 14.0 | 0.0000 | 54 |
| $100^-$ | 7.3 | 10.4 | 26.7 | 77.1 | 2.5 | 0 | 0.0001 | 17340 | 3.3 | 8.7 | 0.9 | 0 | 10.5 | 0.0000 | 594 | 1.6 | 2.6 | 1.0 | 0 | 12.0 | 0.0000 | 91 |
| $200^+$ | 77.2 | 115.1 | 1858.8 | 3600.0 | 215.8 | 2 | 0.0048 | 295087 | 24.0 | 58.1 | 9.3 | 0 | 101.2 | 0.0001 | 938 | 6.1 | 10.3 | 3.1 | 0 | 121.2 | 0.0001 | 91 |
| $200^0$ | 103.5 | 139.9 | 3317.2 | 3600.0 | 1212.5 | 8 | 0.0360 | 581763 | 39.8 | 80.3 | 13.5 | 0 | 143.2 | 0.0001 | 1656 | 7.1 | 11.3 | 4.0 | 0 | 147.0 | 0.0001 | 153 |
| $200^-$ | 72.9 | 108.2 | 3600.0 | 3600.0 | 3600.0 | 10 | 0.0653 | 690213 | 92.2 | 191.3 | 19.2 | 0 | 165.1 | 0.0001 | 4594 | 11.8 | 23.7 | 5.1 | 0 | 120.0 | 0.0001 | 454 |
| $300^+$ | 320.1 | 443.9 | 3600.0 | 3600.0 | 3600.0 | 10 | 0.1080 | 343506 | 714.7 | 2642.8 | 80.0 | 0 | 1034.8 | 0.0001 | 10750 | 51.1 | 118.5 | 14.4 | 0 | 495.0 | 0.0001 | 665 |
| $300^0$ | 326.7 | 425.2 | 3600.0 | 3600.0 | 3600.0 | 10 | 0.1689 | 363285 | 2277.0 | 3600.0 | 539.5 | 4 | 2603.7 | 0.0027 | 50212 | 213.8 | 683.3 | 50.2 | 0 | 639.0 | 0.0001 | 4533 |
| $300^-$ | 305.8 | 407.2 | 3600.0 | 3600.0 | 3600.0 | 10 | 0.1998 | 369726 | 3079.8 | 3600.0 | 487.9 | 8 | 3385.6 | 0.0066 | 78310 | 636.2 | 3600.0 | 43.6 | 1 | 1043.4 | 0.0003 | 16235 |
| $400^+$ | 1095.4 | 1314.5 | 3600.0 | 3600.0 | 3600.0 | 10 | 0.1560 | 221302 | 2714.9 | 3600.0 | 615.1 | 6 | 3810.4 | 0.0043 | 17097 | 393.5 | 1387.6 | 56.1 | 0 | 1708.0 | 0.0001 | 3346 |
| $400^0$ | 989.8 | 1188.7 | 3600.0 | 3600.0 | 3600.0 | 10 | 0.2211 | 218623 | 3213.8 | 3600.0 | 1005.5 | 8 | 4203.7 | 0.0122 | 14928 | 1912.6 | 3600.0 | 79.9 | 4 | 3101.3 | 0.0005 | 20612 |
| $400^-$ | 1046.5 | 1249.4 | 3600.0 | 3600.0 | 3600.0 | 10 | 0.2545 | 212053 | 3564.2 | 3600.0 | 3241.6 | 9 | 4610.7 | 0.0214 | 12729 | 2627.6 | 3600.0 | 154.5 | 7 | 3877.0 | 0.0023 | 32960 |

**Table 2**
Performance of MIQP reformulations on the test set B.

| $n$ | $time_{2r}$ | $time_{1r}$ | (P) | | | | | | (P2R($\theta$)) | | | | | | | ($P_q(u,v,w)$) | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | Time | Max | Min | uns | Gap | Node | Time | Max | Min | uns | Total | Gap | Node | Time | Max | Min | uns | Total | Gap | Node |
| $100^+$ | 10.9 | 14.1 | 2.9 | 10.4 | 1.3 | 0 | 0.0000 | 953 | 1.8 | 3.7 | 1.0 | 0 | 12.7 | 0.0000 | 126 | 1.9 | 2.5 | 1.1 | 0 | 16.0 | 0.0000 | 32 |
| $100^0$ | 11.0 | 14.4 | 4.6 | 11.8 | 0.8 | 0 | 0.0000 | 1779 | 2.1 | 3.6 | 1.3 | 0 | 13.1 | 0.0000 | 145 | 1.6 | 2.7 | 1.1 | 0 | 16.0 | 0.0000 | 45 |
| $100^-$ | 10.2 | 14.7 | 8.8 | 30.2 | 1.4 | 0 | 0.0000 | 3505 | 2.5 | 6.2 | 1.1 | 0 | 12.8 | 0.0000 | 306 | 1.7 | 2.7 | 1.1 | 0 | 16.3 | 0.0000 | 64 |
| $200^+$ | 98.8 | 133.9 | 682.0 | 3600.0 | 41.6 | 1 | 0.0013 | 60411 | 33.0 | 56.0 | 19.3 | 0 | 131.8 | 0.0001 | 1003 | 5.0 | 7.5 | 3.2 | 0 | 139.0 | 0.0001 | 143 |
| $200^0$ | 100.1 | 131.5 | 1728.8 | 3600.0 | 243.3 | 2 | 0.0035 | 138279 | 80.9 | 165.4 | 31.4 | 0 | 181.0 | 0.0001 | 2683 | 11.8 | 27.2 | 5.8 | 0 | 143.3 | 0.0001 | 490 |
| $200^-$ | 98.8 | 124.5 | 2241.6 | 3600.0 | 139.7 | 5 | 0.0113 | 196940 | 174.9 | 652.8 | 16.2 | 0 | 273.7 | 0.0001 | 4260 | 16.8 | 37.8 | 4.3 | 0 | 141.3 | 0.0001 | 840 |
| $300^+$ | 421.4 | 532.2 | 3600.0 | 3600.0 | 3600.0 | 10 | 0.0426 | 145485 | 944.3 | 1735.9 | 152.7 | 0 | 1365.7 | 0.0001 | 10784 | 66.7 | 113.1 | 20.4 | 0 | 598.9 | 0.0001 | 1708 |
| $300^0$ | 428.1 | 559.7 | 3600.0 | 3600.0 | 3600.0 | 10 | 0.0719 | 168636 | 1747.7 | 3600.0 | 361.5 | 2 | 2175.8 | 0.0019 | 21585 | 101.0 | 141.0 | 41.0 | 0 | 660.7 | 0.0001 | 1865 |
| $300^-$ | 465.7 | 559.7 | 3600.0 | 3600.0 | 3600.0 | 10 | 0.0820 | 162665 | 1649.1 | 3600.0 | 277.0 | 2 | 2114.9 | 0.0021 | 20918 | 216.7 | 613.3 | 41.6 | 0 | 776.4 | 0.0001 | 3861 |
| $400^+$ | 1469.0 | 1670.1 | 3600.1 | 3600.0 | 3600.0 | 10 | 0.0745 | 98388 | 2302.9 | 3600.0 | 487.1 | 4 | 3771.9 | 0.0024 | 12096 | 285.2 | 1331.3 | 49.9 | 0 | 1955.3 | 0.0001 | 2378 |
| $400^0$ | 1380.0 | 1735.1 | 3600.1 | 3600.0 | 3600.0 | 10 | 0.1074 | 97926 | 3488.9 | 3600.0 | 2488.0 | 9 | 4868.9 | 0.0072 | 24386 | 261.3 | 427.3 | 68.5 | 0 | 1996.4 | 0.0001 | 2211 |
| $400^-$ | 1435.0 | 1655.6 | 3600.1 | 3600.0 | 3600.0 | 10 | 0.1160 | 89581 | 3521.9 | 3600.0 | 3181.8 | 8 | 4956.9 | 0.0133 | 20435 | 936.1 | 3283.0 | 99.5 | 0 | 2591.7 | 0.0001 | 8205 |

The column "total" under $(P)_q(u, v, w)$ in Table 2 is the summation of "time" and "($time_{1r}$)". The following are the main observations:

- For approaches using (P) and (P2R($\theta$)), the test problems with less diagonally dominant $Q$ in the objective function tend to be harder to solve and requires more computing time. But the approach using ($P_q(u, v, w)$) is less sensitive to this diagonal dominance.
- When the problem size is 100, the performance of (P) is acceptable. After accounting for the SDP calculation time, the three approaches perform more or less the same in terms of the total computing time. When the problem size is larger, the performance or (P) is quite poor compared to the other two approaches[8].
- If the SDP computing time is not accounted for, the reformulation ($P_q(u, v, w)$) performs best on average in nearly all the lines (except for the first line in Table 2). Figs. 1 and 2 compare the time (in logarithmic scale[9]) used by CPLEX in solving different reformulations. Error bars are used to mark the "max" and "min" columns in Tables 1 and 2. The reformulation ($P_q(u, v, w)$) performs better than the other two reformulations on both test sets, even though we use two different approaches to obtain the parameters $(u, v, w)$.
- After accounting for the SDP computing time, the second approach using the reformulation (P2R($\theta$)) performs similar to the third approach using the reformulation ($P_q(u, v, w)$) when the problem size is 100 or 200. When the problem size is 300 or 400, the third approach using ($P_q(u, v, w)$) performs better with less computing time and smaller gaps. This confirms the effectiveness of our new reformulation.

## 7. Conclusions

To tackle the NP-hard quadratic programming problems with on–off constraints, we have generalized the quadratic convex reformulation (QCR) approach in the literature to derive a new mixed-integer quadratic programming (MIQP) reformulation that can be more efficiently solved by standard MIQP solvers. More specifically, while the objective functions in the original problem (P) and the two reformulations (P1($\theta$)) and (P2($\theta$)) only involve a quadratic term in $x$, our new reformulation lifts the original objective function to a quadratic function of both $x$, $y$ and additional auxiliary decision variable $z$ and ensures its convexity. Compared to the conventional QCR approach that requires the added quadratic function to vanish in the entire feasible region, our approach only requires the added quadratic function to vanish in the set of optimal solutions. Under such an increased dimension of freedom in reformulation, we have derived a more general set of quadratic functions that can be added to the objective function by exploiting the structure in the "on/off" inequality constraints. To search for the best quadratic function that can be used to construct the new reformulation, we only need to solve an SDP problem. After obtaining the parameters, the new reformulation can be readily plugged into and solved by any standard MIQP solver. The advantage of the new reformulation is that its continuous relaxation provides a much tighter lower bound than that from the original standard MIQP formulation, thus accelerating the branch-and-bound process in the MIQP solvers. Our computational tests have verified that, when solved in standard MIQP solvers, our new reformulation performs better than the standard formulation and the reformulations in the literature significantly when the number of on/off constraints is large, because more on/off constraints leads to relatively tighter lower bounds from our new reformulation. In a broader sense, our reformulation approach in this paper generalizes the framework of the QCR approach via providing additional mechanism to derive tight and effective reformulations to tackle hard MIQP problems.

[8] In our randomly generated numerical instance, $m = M = n/2$. Thus the number the on/off constraints and other linear constraints grows linear with $n$. If we limit the number of the on/off constraints and other linear constraints, CPLEX could perform well with the formulation (P).

[9] We use a logarithmic scale (base 10) in the y-axis because the normal scale would obscure the $n = 100$ cases.

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