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Mathematics of Operations Research

Publication details, including instructions for authors and subscription information:
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To cite this article:
Rujun Jiang, Duan Li (2022) Exactness Conditions for Semidefinite Programming Relaxations of Generalization of the Extended Trust Region Subproblem. Mathematics of Operations Research

Published online in Articles in Advance 23 Aug 2022

. https://doi.org/10.1287/moor.2022.1305

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MATHEMATICS OF OPERATIONS RESEARCH
Articles in Advance, pp. 1–19
ISSN 0364-765X (print), ISSN 1526-5471 (online)
http://pubsonline.informs.org/journal/moor

Exactness Conditions for Semidefinite Programming Relaxations of Generalization of the Extended Trust Region Subproblem

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Received: March 4, 2019
Revised: May 1, 2020; September 9, 2021
Accepted: July 1, 2022
Published Online in Articles in Advance: August 23, 2022

MSC2020 Classification: Primary: 90C20, 90C22, 90C26; secondary: 90C30

https://doi.org/10.1287/moor.2022.1305

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Abstract. The extended trust region subproblem (ETRS) of minimizing a quadratic objective over the unit ball with additional linear constraints has attracted a lot of attention in the last few years because of its theoretical significance and wide spectra of applications. Several sufficient conditions to guarantee the exactness of its semidefinite programming (SDP) relaxation have been recently developed in the literature. In this paper, we consider a generalization of the extended trust region subproblem (GETRS), in which the unit ball constraint in the ETRS is replaced by a general, possibly nonconvex, quadratic constraint, and the linear constraints are replaced by a conic linear constraint. We derive sufficient conditions for guaranteeing the exactness of the SDP relaxation of the GETRS under mild assumptions. Our main tools are two classes of convex relaxations for the GETRS based on either a simultaneous diagonalization transformation of the quadratic forms or linear combinations of the quadratic forms. We also compare our results to the existing sufficient conditions in the literature. Finally, we extend our results to derive a new sufficient condition for the exactness of the SDP relaxation of general diagonal quadratically constrained quadratic programs, where each quadratic function is associated with a diagonal matrix.

Funding: This work was supported by the National Natural Science Foundation of China [Grants 11801087, 12171100, 72161160340]; Hong Kong Research Grants Council [Grants 14213716 and 14202017]; and Natural Science Foundation of Shanghai [Grant 22ZR1405100].

Keywords: quadratically constrained quadratic programming • extended trust region subproblem • semidefinite programming • second order cone programming

1. Introduction

We consider the following quadratically constrained quadratic program (QCQP),

$$\begin{aligned} \min \quad & f_1(z) := \frac{1}{2} z^T C z + c^T z \\ \text{such that (s.t.)} \quad & f_2(z) := \frac{1}{2} z^T B z + b^T z + e \le 0, \\ & A^T z - d \le_{\mathcal{K}} 0, \end{aligned} \tag{P$_0$}$$

where C and B are $n \times n$ symmetric matrices, not necessary positive semidefinite (p.s.d.); A is an $n \times m$ matrix; c, $b \in \mathbb{R}^n$, $e \in \mathbb{R}$, $d \in \mathbb{R}^m$ and $\le_{\mathcal{K}}$ is the generalized inequality associated with a proper (convex, closed, pointed, and with a nonempty interior) cone $\mathcal{K}$. When $\mathcal{K} = \mathbb{R}^n_+$, that is, the nonnegative orthant, the inequality $A^T z - d \le_{\mathcal{K}} 0$ becomes the usual inequality $A^T z - d \le 0$. Problem (P_0) is nonconvex because both the quadratic objective and the quadratic constraint may be nonconvex. In fact, Problem (P_0) is in general NP-hard even when there is no quadratic constraint (Pardalos [31]). When m is fixed, $\mathcal{K} = \mathbb{R}^n_+$, and the quadratic constraint reduces to a unit ball; Problem (P_0) has been proved to be polynomially solvable (Bienstock and Michalka [5], Hsia and Sheu [16]).

When there is no conic linear constraint and the quadratic constraint $f_2(z) \le 0$ is a unit ball, Problem (P_0) reduces to the classical *trust region subproblem* (TRS). The TRS first arises in the trust region method for unconstrained optimization problems (Conn et al. [10]) and also admits important applications in robust optimization

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(Ben-Tal and Nemirovski [4]). Various methods have been developed to solve the TRS (Hazen and Koren [14], Martinez [26], Moré and Sorensen [28], Rendl and Wolkowicz [33], Ye [39]). When there is no additional conic linear constraint, Problem (P_0) reduces to the *generalized trust region subproblem* (GTRS), which is also a well-studied subject in the literature. Although it is nonconvex, GTRS enjoys hidden convexity and thus can be solved as fast as solving a convex problem, because of the celebrated S-lemma (Yakubovich [37]). Over the past two decades, numerous methods have been developed for solving the GTRS under mild assumptions; see Ben-Tal and Teboulle [3], Feng et al. [12], Moré [27], Stern and Wolkowicz [34], and Sturm and Zhang [35]. Very recently, Ben-Tal and den Hertog [1] demonstrated that the GTRS admits a second order cone programming (SOCP) reformulation when the two quadratic forms are simultaneously diagonalizable (SD); that is, there exists a nonsingular matrix U such that U^TCU and U^TBU both become diagonal matrices, where the superscript T denotes the transpose of a matrix. The finding in Ben-Tal and den Hertog [1] motivated our previous work [23] to fully characterize the GTRS by deriving a necessary condition, using a canonical form of two real symmetric matrices, under which the GTRS is bounded from below, and revealing that the GTRS is SOCP representable under such a condition. Moreover, in another of our earlier work [21], we also derived a new convex reformulation for the GTRS and developed an efficient solution algorithm. We further proved the linear time solvability of the GTRS in terms of its nonzero entries of the matrices in a recent work [22]. When the quadratic constraint $f_2(z) \le 0$ reduces to a unit ball constraint and $\mathcal{K} = \mathbb{R}^n_+$, Problem ($P_0$) is termed the *extended trust region subproblem* (ETRS), which has recently attracted much attention in the literature (Burer and Anstreicher [7], Burer and Yang [8], Fallahi et al. [11], Ho-Nguyen and Kilinç-Karzan [15], Hsia and Sheu [16], Jeyakumar and Li [18], Locatelli [24], Sturm and Zhang [35], Yang and Burer [38], Ye and Zhang [40]). The ETRS is nonconvex, and the semidefinite programming (SDP) relaxation has been a widely used technique for solving the ETRS. However, the SDP relaxation is often not tight enough and consequently only offers a lower bound, even for the case with $m = 1$ (Sturm and Zhang [35]). Jeyakumar and Li [18] first provided the following dimension condition under which the SDP relaxation is exact,

$$\dim \operatorname{Ker}(C - \lambda_{\min}(C) I_n) \ge \dim \operatorname{span}\{a_1, \ldots, a_m\} + 1,$$

where $\lambda_{\min}(C)$ stands for the minimal eigenvalue of C and $[a_1, \ldots, a_m] = A$, and showed its immediate application in robust least squares and a robust SOCP model problem. Hsia and Sheu [16] derived a more general sufficient condition,

$$\operatorname{rank}[C - \lambda_{\min}(C) I_n, a_1, \ldots, a_m] \le n - 1.$$

After that, using the Karush–Kuhn–Tucker (KKT) conditions of the SDP relaxation (in fact, an equivalent SOCP relaxation) of the ETRS, Locatelli [24] presented a more general sufficient condition than Hsia and Sheu [16], which corresponds to the solution conditions of a specific linear system. Meanwhile, Ho-Nguyen and Kilinç-Karzan [15] also developed a sufficient condition for the ETRS by identifying the feasibility of a linear system; moreover, their condition also guarantees a convex reformulation for a variant of the ETRS with general $\mathcal{K}$ beyond $\mathbb{R}^n_+$. In fact, the two conditions in Locatelli [24] and Ho-Nguyen and Kilinç-Karzan [15] are equivalent for the ETRS when $\mathcal{K} = \mathbb{R}^n_+$ as stated in Ho-Nguyen and Kilinç-Karzan [15]. Jeyakumar and Li [19] considered a minimax diagonal QCQP, where the objective function is of the form

$$\max_{1 \le l \le p} \quad \frac{1}{2} x^T C_l x + c_l^T x + \omega_l$$

and all the quadratic forms (including the quadratic constraints) are SD, and demonstrated that if the epigraphical set of all the functions is convex and closed, then the SDP relaxation, which can be further reformulated into an SOCP problem, is exact.

In this paper, we mainly focus on a *generalization* of the ETRS (GETRS) of form (P_0). This kind of generalization has applications in signal processing and financial engineering. For example, Huang and Sidiropoulos [17] proposed a consensus alternating direction method of multipliers to solve QCQPs arisen in signal processing, where the main cost is a subproblem in form of the GTRS. Then we can generalize their method to solve QCQPs with additional conic linear constraints, where the associated subproblem is now in form of the GETRS. Another application comes from an optimal portfolio deleveraging problem with cross-impact (Luo et al. [25]), where the objective is to maximize the equity (a nonconcave quadratic function) subject to a leverage constraint (a nonconvex quadratic function) and a polyhedron constraint. Specifically, the quadratic objective function (equity) refers to the difference between the values of the portfolio and the liability; the quadratic constraint requires that the leverage ratio of liability over equity does not exceed a predetermined bound; and the polyhedron constraint

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contains all requirements for the trading amounts. To the best of our knowledge, the current literature lacks study on efficient solution schemes for the GETRS and the equivalence between the GETRS and its SDP/SOCP relaxation. Our study in this paper is motivated not only by wide applications of the GETRS but also by its theoretical implication to a more general class of QCQPs. The GETRS is much more difficult than the ETRS: The feasible region of the GETRS is no longer compact, and the null space of $C + uB$ in the GETRS is more complicated than that in the ETRS, where u is the corresponding KKT multiplier of constraint $f_2(z) \le 0$. We consider the SDP-based conic programming relaxation (for notational simplicity, we will also call it the SDP relaxation) of Problem (P_0):

$$\begin{aligned} \min \quad & \frac{1}{2} Z \bullet C + c^T z \\ \text{s.t.} \quad & \frac{1}{2} Z \bullet B + b^T z + e \le 0, \\ & A^T z - d \preceq_{\mathcal{K}} 0, \\ & \begin{pmatrix} 1 & z^T \\ z & Z \end{pmatrix} \succeq 0. \end{aligned} \tag{SDP_0}$$

To introduce our investigation of sufficient conditions when the SDP relaxation is exact, we first define the set $I_{PSD} = \{\lambda : C + \lambda B \succeq 0\} \cap \mathbb{R}_+$, where $\mathbb{R}_+$ is the set consisting of all nonnegative real numbers. The set I_{PSD} is in fact an interval (Jiang and Li [21], Moré [27]). Denote the boundary and the interior of a set S by ∂S and $\mathrm{int}(S)$, respectively. Then we have the following possibilities for I_{PSD},

$$I_{PSD} = [\lambda_1, \lambda_2],\ I_{PSD} = [\lambda_1, +\infty),\ I_{PSD} = \{\lambda_1\},\ \text{or}\ \ I_{PSD} = \emptyset,$$

for some $0 \le \lambda_1 < \lambda_2 < +\infty$. In this paper, we will investigate the nontrivial cases, that is, the first three cases. In these cases, the boundary of the interval I_{PSD} can be written as either $\partial I_{PSD} = \{\lambda_1, \lambda_2\}$ or $\partial I_{PSD} = \{\lambda_1\}$. We then develop different sufficient conditions in the following two cases, respectively:

1. The interval $I_{PSD} \ne \emptyset$, and I_{PSD} is not a singleton (i.e., $\mathrm{int}(I_{PSD}) \ne \emptyset$), in which case C and B must be SD (Jiang and Li [21]).
2. The interval I_{PSD} is a singleton, and C and B are SD.

In case 1, to prove the exactness, we derive two different classes of convex relaxations for the GETRS, which are equivalent to the SDP relaxation. The first one is based on simultaneous diagonalization of the quadratic forms; and the second one is based on aggregation of the quadratic forms, where we only need to compute at most two generalized eigenvalues of the quadratic forms. We propose sufficient conditions to guarantee the exactness of the convex relaxations and also the exactness of the SDP relaxation. Moreover, we also extend our conditions for Problem (P_0) with its objective function replaced by a finite maximum of quadratic functions associated with the same matrix, that is, $\max_{1\le l\le p} \frac{1}{2} x^T C x + c_l^T x + \omega_l$. We further obtain a generalization of the celebrated S-lemma under some mild conditions. Our new S-lemma, where two quadratic forms are both general, covers the S-lemma for extended trust region system in Jeyakumar and Li [18], where the quadratic constraint is a strongly convex function, as a special case. In case 2, generalizing the previous conditions, we propose sufficient conditions for the exactness of the SDP relaxation.

We also consider general diagonal QCQPs, where each quadratic function is associated with a diagonal matrix. Burer and Ye [9] recently proposed a new sufficient condition under which the SDP relaxations of diagonal QCQPs are exact. Moreover, they proved that their results can be applied to nondiagonal QCQPs and further show that the SDP relaxations are exact with high probability for a class of randomly generated QCQPs. We compare their sufficient conditions with ours for the GETRS and show the advantages of our conditions. We then propose a new exactness condition for the SDP relaxation of diagonal QCQPs, based on the exactness conditions for the SDP relaxation of the GETRS. It then follows from the same arguments in Burer and Ye [9] that our new sufficient condition can also be applied to nondiagonal QCQPs and the same class of randomly generated QCQPs.

The remainder of the paper is organized as follows. In Section 2, we derive two different classes of convex relaxations for (P_0) under mild assumptions. Based on these relaxations, we propose sufficient conditions for the exactness of the convex relaxations and thus the SDP relaxation, and we also derive a variant of the S-lemma. In Section 3, we compare our sufficient conditions with those in Burer and Ye [9] for the exactness of the SDP

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relaxation of the GETRS and then propose a new sufficient condition for the exactness of SDP relaxations of diagonal QCQPs. We conclude our paper in Section 4.

1.1. Notation

For any index set J, we define A_J as the restriction of matrix A to the rows indexed by J and v_J as the restriction of vector v to the entries indexed by J. We denote by the notation J^C the complementary set of J. We use $\mathrm{Diag}(A)$ and $\mathrm{diag}(a)$ to denote the vector formed by the diagonal entries of matrix A and the diagonal matrix formed by vector a, respectively. The notation $v(\cdot)$ represents the optimal value of problem $(\cdot)$. We use $\mathrm{Null}(A)$ to denote the null space of matrix A. We use $\mathcal{A} \setminus \mathcal{B}$ to denote the relative complement of $\mathcal{B}$ in $\mathcal{A}$, that is, the set whose elements are members of $\mathcal{A}$ but not members of $\mathcal{B}$. We denote by $a_{i:j}$ the vector whose components are entries from i to j of a vector a.

2. Convex Quadratic Relaxations and Exactness Conditions for the GETRS

In this section, we present our main results on exactness conditions for the SDP relaxation of (P_0). We first make some blanket assumptions in Section 2.1. We then consider the case $\mathrm{int}(I_{PSD}) \neq \emptyset$ in Sections 2.2 and 2.3. Our main techniques are based on two classes of convex relaxations of (P_0) that are equivalent to the SDP relaxation of (P_0). The first class is based on simultaneous diagonalization of the quadratic forms and the second class is based on aggregation of the quadratic functions, where at most two generalized eigenvalues of the quadratic forms are required. Our sufficient conditions guarantee the exactness of both convex relaxations and thus the exactness of the SDP relaxation. We also generalize this result for a variant of problem (P_0) with an objective function in a finite maximum form. Then in Section 2.4, we present a variant of the S-lemma under suitable assumptions. Finally in Section 2.5, we extend the previous exactness conditions for the case where I_{PSD} is a singleton, and C and B are SD.

2.1. Blanket Assumptions

We first make the following blanket assumptions throughout Section 2.

Assumption 2.1. *Assume the Slater condition holds for Problem* (SDP_0), *that is, there exist* $Z \in \mathbb{R}^{n\times n}$ *and* $z \in \mathbb{R}^n$ *such that*

$$\begin{cases} \dfrac{1}{2} Z \bullet B + b^T z + e \leq 0,\ A^T z - d \leq_{\mathcal{K}} 0,\ Z \succ zz^T, & \text{if } \mathcal{K} \text{ is a polyhedron,} \\ \dfrac{1}{2} Z \bullet B + b^T z + e \leq 0,\ A^T z - d <_{\mathcal{K}} 0,\ Z \succ zz^T, & \text{if } \mathcal{K} \text{ is not a polyhedron.} \end{cases}$$

Assumption 2.2. *The two matrices C and B are not both p.s.d. matrices.*

Assumption 2.1 is widely used in the SDP literature to ensure strong duality (see, e.g., Burer and Ye [9], Fujie and Kojima [13], Ye and Zhang [40]). Moreover, under Assumption 2.1, if Assumption 2.2 fails, Problem (P_0) is a convex QCQP and admits exact SDP relaxations (see, e.g., Fujie and Kojima [13]).

2.2. Convex Relaxation Based on Simultaneous Diagonalization

In this section, we consider the case where the two matrices in the quadratic forms are SD in (P_0); that is, there exists a nonsingular matrix U such that $U^T C U$ and $U^T B U$ both become diagonal matrices. A specific algorithm to identify two matrices SD or not can be found in Jiang and Li [20]. Then Problem (P_0) can be reformulated, via a change of variables $z = Ux$, as follows,

$$\begin{aligned} \min\quad & \sum_{i=1}^n \frac{1}{2}\delta_i x_i^2 + \sum_{i=1}^n \varepsilon_i x_i \\ \text{s.t.}\quad & \sum_{i=1}^n \frac{1}{2}\alpha_i x_i^2 + \sum_{i=1}^n \beta_i x_i + e \leq 0, \\ & \bar{A}^T x - d \leq_{\mathcal{K}} 0, \end{aligned} \tag{P_{SD}}$$

where $\delta = \mathrm{Diag}(U^T C U)$, $\alpha = \mathrm{Diag}(U^T B U)$, $\varepsilon = U^T c$, $\beta = U^T b$ and $\bar{A} = U^T A$. By invoking augmented variables $y_i = x_i^2$ and relaxing to $y_i \geq x_i^2$, we have the following convex relaxation, which can also be cast as an SOCP problem

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with a conic linear constraint,

$$\begin{aligned} \min\quad & \sum_{i=1}^{n}\frac{1}{2}\delta_i y_i + \sum_{i=1}^{n}\varepsilon_i x_i \\ \text{s.t.}\quad & \sum_{i=1}^{n}\frac{1}{2}\alpha_i y_i + \sum_{i=1}^{n}\beta_i x_i + e \le 0, \\ & \bar{A}^T x - d \le_{\mathcal{K}} 0, \\ & x_i^2 \le y_i,\ i = 1,\dots,n. \end{aligned} \tag{SOCP$_{\text{SD}}$}$$

It is easy to see that (SOCP$_{\text{SD}}$) is equivalent to (SDP$_0$). Thus, we only need to focus on identifying the exactness of (SOCP$_{\text{SD}}$).

It is well known that under the Slater condition (Assumption 2.1), any optimal solution of convex problems must satisfy the KKT conditions (Boyd and Vandenberghe [6]). This fact enables us to find sufficient conditions that guarantee the exactness of the SDP/SOCP relaxation. Let us denote the jth column of matrix $\bar{A}$ by $a^{(j)}$. Then the KKT conditions of the convex Problem (SOCP$_{\text{SD}}$) are given as follows:

$$\begin{aligned} & \frac{1}{2}(\delta_i + u\alpha_i) - w_i = 0, && i = 1,\dots,n, \\ & \varepsilon_i + u\beta_i + \sum_{j=1}^{m} v_j a_i^{(j)} + 2w_i x_i = 0, && i = 1,\dots,n, \\ & \sum_{i=1}^{n}\frac{1}{2}\alpha_i y_i + \sum_{i=1}^{n}\beta_i x_i + e \le 0, && \\ & \bar{A}^T x - d \le_{\mathcal{K}} 0, && \\ & x_i^2 \le y_i, && i = 1,\dots,n, \\ & u\left(\sum_{i=1}^{n}\frac{1}{2}\alpha_i y_i + \sum_{i=1}^{n}\beta_i x_i + e\right) = 0, && \\ & v^T\left(\bar{A}^T x - d\right) = 0, && \\ & w_i(x_i^2 - y_i) = 0, && i = 1,\dots,n, \\ & u, w_i \ge 0, && i = 1,\dots,n, \\ & v \ge_{\mathcal{K}^*} 0, && \end{aligned} \tag{1}$$

where u is the KKT multiplier of the constraint $\sum_{i=1}^{n}\frac{1}{2}\alpha_i y_i + \sum_{i=1}^{n}\beta_i x_i + e \le 0$, v is the KKT multiplier of the conic linear constraint $\bar{A}^T x - d \le_{\mathcal{K}} 0$, $\mathcal{K}^*$ is the dual cone of $\mathcal{K}$, and w_i is the KKT multiplier of the constraint $x_i^2 \le y_i$, $i = 1,\dots,n$.

For any $u \in I_{PSD}$, let us define $J(u) = \{i : \delta_i + u\alpha_i = 0,\ i = 1,\dots,n\}$, that is, the index set corresponding to the null space of $\text{diag}(\delta) + u\,\text{diag}(\alpha)$. We also define $\hat{J} = \{i : \delta_i = \alpha_i = 0, i \in J(u)\}$, that is, the index set corresponding to the common null space of $\text{diag}(\delta)$ and $\text{diag}(\alpha)$. We will use J instead of $J(u)$ for simplicity if it does not cause any confusion. So we have $\bar{A}_J = [a_J^{(1)},\dots,a_J^{(m)}]$. We next show a sufficient condition, which is a generalization of the results in Locatelli [24] and Ho-Nguyen and Kilinç-Karzan [15], to guarantee the exactness of the SDP relaxation of (P$_0$).

Condition 2.1. *The interior of I_{PSD} is nonempty, that is,* $\text{int}(I_{PSD}) \ne \emptyset$. *For any $u \in \partial I_{PSD}$, if $J \backslash \hat{J} \ne \emptyset$, there exists a $z \in \mathbb{R}^{|J|}$ (for notational simplicity we suppose the index set of z is J) such that $z_i \ne 0$ for some $i \in J \backslash \hat{J}$, $\bar{A}_J^T z \le_{\mathcal{K}} 0$ and $(\varepsilon_J + u\beta_J)^T z \le 0$.*

To proceed, we first propose a lemma that helps us understand Condition 2.1.

Lemma 2.1. *Suppose that* $\text{int}(I_{PSD}) \ne \emptyset$ *and Assumption 2.2 holds. If $J \backslash \hat{J} \ne \emptyset$, then we have either $\alpha_i > 0\ \ \forall i \in J \backslash \hat{J}$ or $\alpha_i < 0\ \ \forall i \in J \backslash \hat{J}$.*

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Proof. If $\alpha_i = 0$ for some $i \in J\backslash\hat{J}$, then we must have $\delta_i = -u\alpha_i = 0$, contradicting $i \in J\backslash\hat{J}$. Hence, we know that $\alpha_i \neq 0$ for $i \in J\backslash\hat{J}$. First consider the case $u \neq 0$. For any $i, j \in J\backslash\hat{J}$, we must have $\delta_i = -u\alpha_i$ and $\delta_j = -u\alpha_j$. If α_i and α_j have different signs, then we have $I_{PSD} = \{u\}$, contradicting $\mathrm{int}(I_{PSD}) \neq \emptyset$. Now consider the case $u = 0$. In this case, we have $C = C + 0B \succeq 0$ and thus $\alpha_i > 0$ for all $i \in J\backslash\hat{J}$ if $J\backslash\hat{J} \neq \emptyset$, due to $\mathrm{int}(I_{PSD}) \neq \emptyset$. This completes the proof. □

Now we are ready to present our main result in this subsection.

Theorem 2.1. *Suppose that Assumptions 2.1 and 2.2 hold and Condition 2.1 holds. Suppose further that there exists an optimal solution for* ($\mathrm{SOCP_{SD}}$). *Then* $v(\mathrm{P_0}) = v(\mathrm{SOCP_{SD}}) = v(\mathrm{SDP_0})$ *and the optimal value of* ($\mathrm{P_0}$) *is attained.*

Proof. Recall that ($\mathrm{P_0}$) is equivalent to ($\mathrm{P_{SD}}$) via a change of variables. Because of the Slater condition, every optimal solution of ($\mathrm{SOCP_{SD}}$) must be a KKT point, denoted by $(u^*, v^*, w^*, x^*, y^*)$, of system (1). As ($\mathrm{SOCP_{SD}}$) is a relaxation of ($\mathrm{P_{SD}}$), we always have $v(\mathrm{SOCP_{SD}}) \le v(\mathrm{P_{SD}})$. We will show that there exists an optimal solution $(\bar{x}, \bar{y})$ of ($\mathrm{SOCP_{SD}}$) satisfying $\bar{x}_i^2 = \bar{y}_i$, $i = 1, \ldots, n$, which means $\bar{x}$ is also an feasible solution of ($\mathrm{P_{SD}}$) with an objective value equal to ($\mathrm{SOCP_{SD}}$). Hence, together with $v(\mathrm{SOCP_{SD}}) \le v(\mathrm{P_{SD}})$, we conclude that $\bar{x}$ is also an optimal solution of ($\mathrm{P_{SD}}$) and $v(\mathrm{P_{SD}}) = v(\mathrm{SOCP_{SD}})$.

If $u^* \in \mathrm{int}(I_{PSD})$, then we have $w_i^* = \frac{1}{2}(\delta_i + u^*\alpha_i) > 0$, for all $i \notin \hat{J}$. By the complementary slackness $w_i^*((x_i^*)^2 - y_i^*) = 0$, we have $(x_i^*)^2 = y_i^*$, for all $i \notin \hat{J}$. Hence, noting that $\delta_i = \alpha_i = 0, i \in \hat{J}$, we conclude that x^* is an optimal solution of Problem ($\mathrm{P_{SD}}$) and thus $v(\mathrm{P_{SD}}) = v(\mathrm{SOCP_{SD}})$.

Next we consider the case $u^* \in \partial I_{PSD}$. From the complementary slackness $w_i^*((x_i^*)^2 - y_i^*) = 0$, we know that $(x_i^*)^2 = y_i^*$, for all $i \notin J$. If $J\backslash\hat{J} = \emptyset$, because of $\delta_i = \alpha_i = 0, i \in \hat{J}$, we conclude that x^* is an optimal solution of Problem ($\mathrm{P_{SD}}$) and thus $v(\mathrm{P_{SD}}) = v(\mathrm{SOCP_{SD}})$. Now consider $J\backslash\hat{J} \neq \emptyset$. Suppose that $(x_i^*)^2 < y_i^*$ holds for at least one index in $J\backslash\hat{J}$ for otherwise x^* is an optimal solution of ($\mathrm{P_{SD}}$) and $v(\mathrm{P_{SD}}) = v(\mathrm{SOCP_{SD}})$. Let $z \in \mathbb{R}^{|J|}$ satisfy Condition 2.1. First consider the case $\sum_{i\in J} \frac{1}{2}\alpha_i z_i^2 > 0$. Because of Lemma 2.1, we must have $\alpha_i > 0$ for all $i \in J\backslash\hat{J}$. Let θ^* be a nonnegative solution of the quadratic equation

$$\sum_{i\in J} \frac{1}{2}\alpha_i(x_i^* + \theta z_i)^2 + \beta_i(x_i^* + \theta z_i) = \sum_{i\in J} \frac{1}{2}\alpha_i y_i^* + \beta_i x_i^*, \tag{2}$$

or equivalently

$$\sum_{i\in J} \frac{1}{2}\alpha_i z_i^2 \theta^2 + \left(\sum_{i\in J} \alpha_i x_i^* z_i + \beta_i z_i\right)\theta + \sum_{i\in J} \frac{1}{2}\alpha_i((x_i^*)^2 - y_i^*) = 0, \tag{3}$$

where such θ^* exists because the quadratic function (3) is nonpositive at $\theta = 0$ and positive if θ is sufficiently large because of $\sum_{i\in J} \frac{1}{2}\alpha_i z_i^2 > 0$. Hence, setting

$$\begin{cases} (\bar{x}_i, \bar{y}_i) = (x_i^* + \theta^* z_i, (x_i^* + \theta^* z_i)^2), & \forall i \in J \\ (\bar{x}_i, \bar{y}_i) = (x_i^*, y_i^*), & \forall i \in J^C \end{cases}$$

gives a new feasible solution of ($\mathrm{SOCP_{SD}}$) with $\bar{x}_i^2 - \bar{y}_i = 0$ for $i = 1, \ldots, n$. Then we obtain that the objective value at the new solution does not increase because

$$\begin{aligned}
\sum_{i\in J} \frac{1}{2}\delta_i \bar{x}_i^2 + \varepsilon_i \bar{x}_i &= \sum_{i\in J} \frac{1}{2}\delta_i (x_i^* + \theta^* z_i)^2 + \varepsilon_i (x_i^* + \theta^* z_i) \\
&= \sum_{i\in J} (\varepsilon_i + u^*\beta_i)(x_i^* + \theta^* z_i) - u^*\left[\frac{1}{2}\alpha_i (x_i^* + \theta^* z_i)^2 + \beta_i (x_i^* + \theta^* z_i)\right] \\
&= \sum_{i\in J} (\varepsilon_i + u^*\beta_i)(x_i^* + \theta^* z_i) - u^*\left[\frac{1}{2}\alpha_i y_i^* + \beta_i x_i^*\right] \\
&= \sum_{i\in J} \frac{1}{2}\delta_i y_i^* + \varepsilon_i x_i^* + \theta^*(\varepsilon_i + u^*\beta_i) z_i \\
&\le \sum_{i\in J} \frac{1}{2}\delta_i y_i^* + \varepsilon_i x_i^*,
\end{aligned}$$

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where the second and fourth equalities are due to $\delta_i + u^*\alpha_i = 0,\ \forall i \in J$; the third equality is due to (2); and the inequality is due to Condition 2.1 and $\theta^* \geq 0$. Therefore, $\bar{x}$ is an optimal solution of Problem ($\mathrm{P_{SD}}$) and thus $v(\mathrm{P_{SD}}) = v(\mathrm{SOCP_{SD}})$.

The case $\sum_{i\in J}\frac{1}{2}\alpha_i z_i^2 < 0$ can be proved in a symmetric way.

We claim that the remaining case of $\sum_{i\in J}\frac{1}{2}\alpha_i z_i^2 = 0$ cannot occur. Indeed, if this is the case, we then must have $\alpha_i z_i = 0\ \ \forall i \in J\backslash\hat{J}$. This, together with Lemma 2.1, further implies $z_i = 0\ \ \forall i \in J\backslash\hat{J}$, contradicting Condition 2.1.

From the above analysis, we obtain that either (x^*, y^*) with $(x_i^*)^2 = y_i^*, i \notin \hat{J}$ or $(\bar{x}, \bar{y})$ with $\bar{x}_i^2 = \bar{y}_i, i \notin \hat{J}$ is an optimal solution of ($\mathrm{SOCP_{SD}}$). Hence, we know that the optimal solution of Problem ($\mathrm{P_{SD}}$) is attained by either x^* or $\bar{x}$, which implies that the optimal value of ($\mathrm{P_0}$) is attained. □

Remark 2.1. *The assumption that there exists an optimal solution for* ($\mathrm{SOCP_{SD}}$) *holds under mild conditions, for example, under the assumptions in Theorem 2.1 and in addition $\hat{J} = \emptyset$. Indeed, the Slater condition (Assumption 2.1) implies that* ($\mathrm{SOCP_{SD}}$) *is feasible; hence, the Lagrangian dual of* ($\mathrm{SOCP_{SD}}$) *is bounded from weak duality. The conditions* $\mathrm{int}(I_{PSD}) \neq \emptyset$ *and $\hat{J} = \emptyset$ imply that the Lagrangian dual of* ($\mathrm{SOCP_{SD}}$) *satisfies the Slater condition. Hence, strong duality holds between* ($\mathrm{SOCP_{SD}}$) *and its Lagrangian dual, and the optimal solution of* ($\mathrm{SOCP_{SD}}$) *is attained (see, e.g., Ben-Tal and Nemirovski [2]).*

If we only care about the exactness of the basic SDP relaxation ($\mathrm{SDP_0}$), then we only need the check the following condition, which is equivalent to Condition 2.1 but in the original space.

Condition 2.2. *The interior of I_{PSD} is nonempty. For any $u \in \partial I_{PSD}$, if* $\mathrm{Null}(C+uB)\backslash(\,(\mathrm{Null}(C)\cap\mathrm{Null}(B)\,) \neq \emptyset$, *there exists a $y \in \mathbb{R}^n$ such that* $y \notin \mathrm{Null}(C)\cap\mathrm{Null}(B)$, $(C+uB)y = 0$, $A^T y \leq_{\mathcal{K}} 0$ *and* $(c+ub)^T y \leq 0$.

We remark that $y \notin \mathrm{Null}(C)\cap\mathrm{Null}(B)$ implicitly implies $y \neq 0$ as we always have $0 \in \mathrm{Null}(C)\cap\mathrm{Null}(B)$, and that $y \notin \mathrm{Null}(C)\cap\mathrm{Null}(B)$ in Condition 2.2 is equivalent to $z_i \neq 0$ for some $i \in J\backslash\hat{J}$ in Condition 2.1. When $\mathrm{Null}(C)\cap\mathrm{Null}(B) = \{0\}$ and there is no conic constraint in ($\mathrm{P_{SD}}$) (i.e., the problem reduces to the GTRS), the condition $\mathrm{int}(I_{PSD}) \neq \emptyset$ is also known as the regular case (Moré [27], Stern and Wolkowicz [34]) or dual Slater condition (Ye and Zhang [40]). We also remark that Condition 2.2 can be checked by computing at most two generalized eigenvalues (the endpoints of I_{PSD}) and solving a conic linear system with dimension $|J|$.

2.3. Alternative Convex Relaxations Based on Aggregation of Quadratic Functions

In this subsection, we first propose two convex relaxations for ($\mathrm{P_0}$), which are equivalent to the SDP relaxation, based on aggregation of the objective function and quadratic constraint, and then show the exactness of the two convex relaxations (and thus the SDP relaxation) under Condition 2.2. Particularly, we consider the following epigraph-based reformulation of ($\mathrm{P_0}$),

$$\begin{aligned} \min\ \ & t \\ \text{s.t.}\ \ & f_1(z) \leq t, \\ & f_2(z) \leq 0, \\ & A^T z - d \leq_{\mathcal{K}} 0. \end{aligned} \tag{P_0'}$$

Recall that in this section we consider the case $\mathrm{int}(I_{PSD}) \neq \emptyset$. When B is not p.s.d., we have that $I_{PSD} = [\lambda_1, \lambda_2]$, where both λ_1 and λ_2 are finite and $\lambda_1 < \lambda_2$. In this case, by aggregating the quadratic constraints, we consider the following convex relaxation for ($\mathrm{P_0}$),

$$\begin{aligned} \min\ \ & t \\ \text{s.t.}\ \ & h_1(z) \leq t, \\ & h_2(z) \leq t, \\ & A^T z - d \leq_{\mathcal{K}} 0, \end{aligned} \tag{P_1}$$

where $h_1(z) = f_1(z) + \lambda_1 f_2(z)$ and $h_2(z) = f_1(z) + \lambda_2 f_2(z)$. When B is p.s.d., we have that $I_{PSD} = [\lambda_1, +\infty)$ and λ_1 is finite. In this case, we consider the following convex relaxation,

$$\begin{aligned} \min\ \ & t \\ \text{s.t.}\ \ & h_3(z) \leq t, \\ & f_2(z) \leq 0, \\ & A^T z - d \leq_{\mathcal{K}} 0, \end{aligned} \tag{P_2}$$

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where $h_3(z) = f_1(z) + \lambda_1 f_2(z)$.

We prove in the following theorem that (P_1) and (P_2) are exact convex relaxations for (P_0) under suitable conditions.

Theorem 2.2. *Suppose Assumption 2.2 and Condition 2.2 hold.*

1. *If B is not p.s.d., and there exists an optimal solution of* (P_1)*, then the convex relaxation* (P_1) *is exact.*
2. *Otherwise if B is p.s.d., and there exists an optimal solution of* (P_2)*, then the convex relaxation* (P_2) *is exact.*

In both cases, the optimal value of (P_0) *is attained.*

Proof. First consider the case where B is not p.s.d. Because of $\mathrm{int}(I_{PSD}) \neq \emptyset$, we have $0 \le \lambda_1 < \lambda_2 < +\infty$. As ($\mathrm{P}_1$) is a relaxation of ($\mathrm{P}_0'$) and ($\mathrm{P}_0'$) is equivalent to ($\mathrm{P}_0$), we always have $v(\mathrm{P}_1) \le v(\mathrm{P}_0)$. In the following, we will prove that for any optimal solution of (P_1), we can construct a feasible solution of (P_0) with the same objective value, which is thus optimal to (P_0) because $v(\mathrm{P}_1) \le v(\mathrm{P}_0)$. Define $Q_i = C + \lambda_i B$ and $p_i = c + \lambda_i b$ for $i = 1, 2$.

Now suppose (z^*, t^*) is an optimal solution of (P_1). First consider the case where $h_1(z^*) = t^*$. This yields $h_1(z^*) = t^* \ge h_2(z^*)$. Hence, from the definitions of h_1 and h_2, we have $(\lambda_2 - \lambda_1) f_2(z^*) \le 0$, which further implies $f_2(z^*) \le 0$ because of $\lambda_1 < \lambda_2$. Thus, z^* is feasible to (P_0). Also note that $f_1(z^*) = t^* - \lambda_1 f_2(z^*)$. If further either $f_2(z^*) = 0$ or $\lambda_1 = 0$, we then have $f_1(z^*) = t^*$, which, together with the feasibility of z^*, implies that z^* is an optimal solution for (P_0). Otherwise we must have $\lambda_1 > 0$ and $f_2(z^*) < 0$. In this case, we know from $\lambda_2 - \lambda_1 > 0$ and $f_2(z^*) < 0$ that

$$h_2(z^*) = h_1(z^*) + (\lambda_2 - \lambda_1) f_2(z^*) < h_1(z^*) = t^*$$

and from $\lambda_1 > 0$ that $\mathrm{Null}(Q_1) \backslash ((\mathrm{Null}(C) \cap \mathrm{Null}(B)) \neq \emptyset$. Because $\mathrm{int}(I_{PSD}) \neq \emptyset$, we have

$$s^T (Q_1 + Q_2) s = 2s\left(C + \frac{\lambda_1 + \lambda_2}{2} B\right) s > 0 \quad \forall s \notin \mathrm{Null}(C) \cap \mathrm{Null}(B). \tag{4}$$

By Condition 2.2, there exists a nonzero y such that

$$Q_1 y = 0, \; y \notin \mathrm{Null(C)} \cap \mathrm{Null}(B), \; A^T y \le_{\mathcal{K}} 0, \quad \text{and } (c + \lambda_1 b)^T y \le 0.$$

Substituting y into (4) implies $y^T Q_2 y > 0$. Now let us consider the quadratic equation

$$h_2(z^* + \theta y) - t^* = \frac{1}{2} y^T Q_2 y \theta^2 + (y^T Q_2 z^* + p_2^T y)\theta + h_2(z^*) - t^* = 0. \tag{5}$$

There exists one positive root θ^* for (5) because of $y^T Q_2 y > 0$ and $h_2(z^*) - t < 0$. Let $\tilde{z} = z^* + \epsilon \theta^* y$. From the optimality of (z^*, t^*) to (P_1), we must have $(c + \lambda_1 b)^T y = 0$ for otherwise we have $h_2(\tilde{z}) < t^*$ and

$$h_1(\tilde{z}) = h_1(z^*) + \epsilon \theta^* (c + \lambda_1 b)^T y < h_1(z^*) = t^*$$

for $\epsilon = \rho \frac{t^* - h_1(z^*)}{\theta^* (c + \lambda_1 b)^T y}$ with $\rho \in (0, 1)$, contradicting the optimality of (z^*, t^*). Then letting $\bar{z} = z^* + \theta^* y$, we have $h_2(\bar{z}) = t^*$ and $h_1(\bar{z}) = h_1(z^*) + \theta^* (c + \lambda_1 b) y = t^*$. This, together with the definition of h_1 and h_2, implies that $f_2(\bar{z}) = 0$ and $f_1(\bar{z}) = t^*$. So $\bar{z}$ is a feasible solution to (P_0) with objective value t^*. This implies that $\bar{z}$ is an optimal solution of (P_0) and $v(\mathrm{P}_0) = v(\mathrm{P}_1)$.

Next we consider the case where $h_1(z^*) < t^*$ and $h_2(z^*) = t^*$. Note that we always have $\mathrm{Null}(Q_2) \backslash (\mathrm{Null}(C) \cap \mathrm{Null}(B)) \neq \emptyset$. It follows from Condition 2.2 that there exists a nonzero y such that

$$Q_2 y = 0, \; y \notin \mathrm{Null}(C) \cap \mathrm{Null}(B), \;\; A^T y \le_{\mathcal{K}} 0 \text{ and } (c + \lambda_2 b)^T y \le 0.$$

Similar to the previous case, we have $y^T Q_1 y > 0$. Consider the quadratic function

$$h_1(z^* + \theta y) - t^* = \frac{1}{2} y^T Q_1 y \theta^2 + (y^T Q_1 z^* + p_2^T y)\theta + h_1(z^*) - t^* = 0. \tag{6}$$

With a similar analysis to the previous case, letting θ^* be a positive root of Equation (6) and $\bar{z} = z^* + \theta^* y$, we have $h_2(\bar{z}) = t^*$ and $h_1(\bar{z}) = t^*$. This again implies that $(\bar{z}, t^*)$ is an optimal solution of (P_0) and $v(\mathrm{P}_0) = v(\mathrm{P}_1)$.

The case where B is p.s.d. can be proved with similar techniques. We just give a sketch of the proof. Suppose (z^*, t^*) is an optimal solution of (P_2). Recall that $h_3(z^*) = f_1(z^*) + \lambda_1 f_2(z^*) = t^*$. Note also that $\lambda_1 = 0$ implies that $C \succeq 0$, which contradicts Assumption 2.2. Hence, we must have $\lambda_1 > 0$. If $f_2(z^*) = 0$, it is easy to see that $f_1(z^*) = t^*$ and thus z^* is an optimal solution of (P_0) with objective value t^*. If $f_2(z^*) < 0$, letting y be a vector satisfying Condition 2.2, we have $y^T B y \neq 0$ for otherwise $(C + \lambda_1 B) y = 0$, together with the positive semidefiniteness of B, implies that $Cy = 0$, contradicting $y \notin \mathrm{Null}(C) \cap \mathrm{Null}(B)$. Hence, we have $y^T B y > 0$ as B is p.s.d. Consider the quadratic

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equation

$$f_2(z^* + \theta y) = \frac{1}{2} y^T B y \theta^2 + b^T y \theta + f_2(z^*) = 0. \tag{7}$$

With a similar analysis to the previous case, letting θ^* be a positive root of Equation (7) and $\bar{z} = z^* + \theta^* y$, we have $f_2(\bar{z}) = 0$ and $h_1(\bar{z}) = t^*$. This implies that $\bar{z}$ is an optimal solution of (P_0) and $v(\mathrm{P}_0) = v(\mathrm{P}_1)$.

From the above constructions, we see that the optimal value of (P_0) is attained. □

Note that in Theorem 2.2, we do not require the Slater condition (Assumption 2.1). The above proof in fact also gives a method to recover an optimal solution of (P_0) from an optimal solution of (P_1) or (P_2). The proof also shows us a geometric interpretation of Condition 2.2; that is, Condition 2.2 indicates the role of ∂I_{PSD} and the existence of a feasible nonincreasing direction for (P_1) and (P_2) for the points that are optimal to (P_1) or (P_2) but not optimal to (P_0).

Remark 2.2. *When there are no conic linear constraints in* (P_0)*, the convex relaxations* (P_1) *and* (P_2) *reduce to the reformulations for the GTRS in the authors' previous work (Jiang and Li* [21, *theorem* 2.10]). *In the GTRS case, the relaxations are always exact. However, in the GETRS, the relaxation may not be exact when Condition* 2.2 *fails. It is obvious that Condition* 2.2 *automatically holds for the GTRS.*

Remark 2.3. *We also remark that in Theorem* 2.2*, as indicated in our proof, the convex relaxations are still exact if* $\mathcal{K}$ *is not pointed or with a nonempty interior, and* $A^T z - d \le_{\mathcal{K}} 0$ *is replaced by* $d - A^T z \in \mathcal{K}$*. In this case, our results reduce to theorem* 2.4 *in Ho-Nguyen and Kilinç-Karzan* [15] *if the quadratic constraint of* (P_0) *reduces to a unit ball. Moreover, if we have* $\mathcal{K} = \mathbb{R}^n_+$ *in addition, then our results reduce to the results in Locatelli* [24].

To formulate (P_1) and (P_2), it only requires to compute at most two generalized eigenvalues of a matrix pencil, that is, the endpoints of I_{PSD}, which costs approximately $O(n^2)$ time, whereas to formulate the SOCP relaxation ($\mathrm{SOCP}_{\mathrm{SD}}$), it requires a simultaneous diagonalization transformation, which costs $O(n^3)$ time in general. See Jiang and Li [21, section 2.1] for a detailed discussion on computation of the endpoints of I_{PSD}. Hence, for large-scale problems, it costs less time to formulate (P_1) or (P_2) than ($\mathrm{SOCP}_{\mathrm{SD}}$). Note that ($\mathrm{P}_1$), ($\mathrm{P}_2$), and ($\mathrm{SOCP}_{\mathrm{SD}}$) can be solved by interior point methods as they are all convex quadratic problems with additional conic linear constraint [30]. Furthermore, using a similar minimax reformulation of the GTRS as in Jiang and Li [21], we may use fast first order methods to solve an equivalent minimax reformulation of Problem (P_1) when the problem dimension is high and interior point methods are prohibited. Specifically, from the structure of problem (P_1), we know that (P_1) is equivalent to the following problem,

$$\begin{aligned} \min_z \quad & \max\{h_1(z), h_2(z)\} \\ \text{s.t.} \quad & A^T z - d \le_{\mathcal{K}} 0. \end{aligned} \tag{P_3}$$

When the projection onto the set $\{z : A^T z - d \le_{\mathcal{K}} 0\}$ is easy to compute, Problem (P_3) can be solved efficiently by many first order methods, for example, the (projected) Nesterov's accelerated gradient method for minimax optimization problems (Nesterov [29], Wang and Kilinç-Karzan [36]) and the steepest descent method (by adding a projection in each iteration) in Jiang and Li [21].

Example 2.1. *This example shows that there may be a positive gap between the optimal values of the GETRS and its SDP/SOCP relaxation when Condition* 2.1 *(or equivalently Condition* 2.2*) fails,*

$$\begin{aligned} \min \quad & 2x_1^2 - x_2^2 - 4x_1 \\ \text{s.t.} \quad & -x_1^2 + x_2^2 \le 1, \\ & x_2 \le 1, \\ & -x_2 \le -1. \end{aligned} \tag{8}$$

In this problem, we have $I_{PSD} = [1, 2]$ *and* $\mathrm{int}(I_{PSD}) \neq \emptyset$*. One may check that Condition* 2.1 *is not fulfilled because for* $u = 1$*, there is no* y *such that* $y \notin \mathrm{Null}(C) \cap \mathrm{Null}(B)$, $(C + B)y = 0$, $A^T y \le 0$ *and* $(c + b)^T y \le 0$*, or equivalently,* $y \neq 0$, $y_1 = 0$, $y_2 = 0$*, and* $-4y_1 \le 0$*. Problem* (8) *is further equivalent to, because of* $x_2 = 1$,

$$\min\ 2x_1^2 - 4x_1 - 1, \ \text{ s.t. } \ x_1^2 \ge 0.$$

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The optimal solution is $x_1 = 1$ and the associated optimal value –3. Hence, the optimal solution of (8) is $x = (1,1)^T$ and the associated optimal value is –3. The SDP relaxation of (8) is

$$\begin{aligned} \min \quad & 2X_{11} - X_{22} - 4x_1 \\ \text{s.t.} \quad & -X_{11} + X_{22} \le 1, \\ & x_2 \le 1, \\ & -x_2 \le -1, \\ & \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0. \end{aligned} \tag{9}$$

We can check that an optimal solution of the SDP relaxation (9) is (X^*, x^*) with $X^* = \begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix}$ and $x^* = (2,1)^T$. Hence, there is a positive gap between Problem (8) and its SDP relaxation.

Motivated by the results in Jeyakumar and Li [19], we consider a natural extension of Condition 2.2 for the case where the objective function is a finite maximum of quadratic functions with the same Hessians (and $\mathcal{K} = \mathbb{R}^n$ for simplicity). That is, we consider the problem

$$\begin{aligned} \min \quad & \tilde{f}_1(z) := \max_{l=1,\ldots,p} \frac{1}{2} z^T C z + c_l^T z + \omega_l \\ \text{s.t.} \quad & f_2(z) := \frac{1}{2} z^T B z + b^T z + e \le 0, \\ & A^T z - d \le 0. \end{aligned} \tag{MP$_0$}$$

Now if B is not p.s.d., and $I_{PSD} = [\lambda_1, \lambda_2]$, where both λ_1 and λ_2 are finite, and $\lambda_1 < \lambda_2$, we have the following convex relaxation,

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & \tilde{h}_1(z) \le t, \\ & \tilde{h}_2(z) \le t, \\ & A^T z - d \le 0, \end{aligned} \tag{MP$_1$}$$

where $\tilde{h}_1(z) = \tilde{f}_1(z) + \lambda_1 f_2(z)$ and $\tilde{h}_2(z) = \tilde{f}_1(z) + \lambda_2 f_2(z)$. When B is p.s.d., and $I_{PSD} = [\lambda_1, +\infty)$ and λ_1 is finite, we have the following convex relaxation,

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & \tilde{h}_3(z) \le t, \\ & f_2(z) \le 0, \\ & A^T z - d \le 0, \end{aligned} \tag{MP$_2$}$$

where $\tilde{h}_3(z) = \tilde{f}_1(z) + \lambda_1 f_2(z)$. We now extend our exactness condition for Problem (MP$_0$).

Condition 2.3. *The interior of I_{PSD} is nonempty. For any $u \in \partial I_{PSD}$, if $\mathrm{Null}(C + uB) \setminus ((\mathrm{Null}(C) \cap \mathrm{Null}(B)) \neq \emptyset$, there exists a $y \in \mathbb{R}^n$ such that $y \notin \mathrm{Null}(C) \cap \mathrm{Null}(B)$, $(C + uB)y = 0$, $A^T y \le 0$ and $(c_l + ub)^T y \le 0$, $l = 1, \ldots, p$.*

Similar to Theorem 2.2, we can show that the convex quadratic relaxation (MP$_1$) or (MP$_2$) is exact.

Theorem 2.3. *Suppose Assumption 2.2 and Condition 2.3 hold.*

1. *If B is not p.s.d., and there exists an optimal solution of (MP$_1$), then the convex relaxation (MP$_1$) is exact.*
2. *Otherwise if B is p.s.d., and there exists an optimal solution of (MP$_2$), then the convex relaxation (MP$_2$) is exact.*

In both cases, the optimal solution of (MP$_0$) is attained.

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Proof. The proof is almost the same with that for Theorem 2.2 except that we need to handle the finite maximum objective function

$$\tilde{f}_1(z) = \max_{l=1,\ldots,p} \frac{1}{2} z^T Cz + c_l^T z + \omega_l = \frac{1}{2} z^T Cz + \max_{l=1,\ldots,p} c_l^T z + \omega_l.$$

Then all the proof follows almost verbatim from that of Theorem 2.2 except that we need some modification when dealing with Equations (5), (6), and (7). We only give an explanation for Equation (5), which now becomes

$$\tilde{h}_2(z^* + \theta y) - t^* = \frac{1}{2}(z^* + \theta y)^T Q_2 (z^* + \theta y) + \max_{1\le l\le p}\{(c_l + ub)^T (z^* + \theta y) + \omega_l + ub\} - t^* = 0.$$

Note that $\tilde{h}_2(z^* + \theta y) - t^* < 0$ for $\theta = 0$ and $\tilde{h}_2(z^* + \theta y) - t^* \to \infty$ as $\theta \to \infty$, because of $y^T Q_2 y > 0$. We conclude that there is a positive root θ^* for $\tilde{h}_2(z^* + \theta y) - t^* = 0$. □

Now let us compare Condition 2.3 with the condition in Jeyakumar and Li [19] (restricted to Problem (MP_0)), which depends on the epigraphical set,

$$E(\phi_1, \ldots, \phi_p, \psi, \varphi_1, \ldots, \varphi_m) = \Big\{(\phi, \psi, \varphi) \in \mathbb{R}^p \times \mathbb{R} \times \mathbb{R}^m : \exists x \in \mathbb{R}^n \text{ such that } \frac{1}{2} z^T Cz + c_l^T z \le \phi_l,$$
$$l = 1, \ldots, p,\ \frac{1}{2} z^T Bz + b^T z + e \le \psi, \text{ and } A^T z - d \le \varphi\Big\}.$$

Though the paper Jeyakumar and Li [19] adopts an SOCP formulation using the SD condition for the matrices, its SOCP relaxation is equivalent to the basic SDP relaxation and thus equivalent to the convex quadratic relaxation (MP_1) or (MP_2).

Theorem 2.4 (theorem 2.1 in Jeyakumar and Li [19]). *Suppose the epigraphical set is a closed convex set, and $v(\mathrm{MP}_0) > -\infty$. Then the basic SDP relaxation is exact.*

However, it is in general difficult to verify if the epigraphical set of given quadratic and linear functions is convex or not. In section 4 of Jeyakumar and Li [19], the authors considered Problem (MP_0) with $B = I$ and derived some abstract condition.

Lemma 2.2 (lemma 4.1 in Jeyakumar and Li [19]). *Suppose that C is not p.s.d. and $B = I_n$ in Problem (MP_0). Let*

$$D = \{y \in \mathbb{R}^{m+1} : f_2(z) \le y_1, A^T z - d \le y_{2:m+1}, \text{ for some } z \in \mathbb{R}^n\}.$$

Suppose that, for each $y \in D$, the convex minimization problem

$$\min_{z\in\mathbb{R}^n} \{h_3(z) : f_2(z) \le y_1, A^T z - d \le y_{2:m+1}, \text{for some } z \in \mathbb{R}^n\} \tag{10}$$

attains its minimum at some $\bar{z} \in \mathbb{R}^n$ with $f_2(\bar{z}) = y_1$. Then the basic SDP relaxation is exact.

Note that the condition in Lemma 2.2 is still too abstract to verify. However, we can show that Condition 2.3 can imply that for each $y \in D$, the convex minimization Problem (10) attains its minimum at some $\bar{z} \in \mathbb{R}^n$ with $f_2(\bar{z}) = y_1$. From our proofs for Theorem 2.3, we can see that Condition 2.3 shows that the convex minimization Problem (MP_2) attains its minimum at some $\bar{z} \in \mathbb{R}^n$ with $f_2(\bar{z}) = 0$ when $y_1 = 0$. Because of the homogeneousness of our condition and using the proof in Theorem 2.3, one can show that the convex minimization Problem (10) attains its minimum at some $\bar{z} \in \mathbb{R}^n$ with $f_2(\bar{z}) = y_1$ for each $y \in D$. In other words, Condition 2.3 gives a condition that is easy to verify for Lemma 2.2.

As the condition in Lemma 2.2 is too difficult to verify, Jeyakumar and Li [19] further derived a dimension condition expressed in the original data of the problem to guarantee exact SDP relaxation for (MP_0).

Theorem 2.5 (theorem 4.1 in Jeyakumar and Li [19]). *Define $Q = (C - \lambda_{\min}(C)I_n, A^T) \in \mathbb{R}^{(n+m)\times n}$. Suppose that $\min(\mathrm{MP}_0) > -\infty$ and $\dim(\mathrm{Ker}Q) \ge p$, where $\mathrm{Ker}Q$ is the kernel of Q, that is, $\mathrm{Ker}Q = \{x : Qx = 0\}$. Then the SDP relaxation is exact.*

We now show that Condition 2.3 contains the condition in Theorem 2.5 as a special case. First note that as there is a ball constraint, we always have $v(\mathrm{MP}_0) > -\infty$ and there exists an optimal solution for (MP_2) when (MP_2) is

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feasible. Indeed, if dim (Ker$Q$) $\geq p$, then there exists a vector $y \in \text{Ker}Q$ such that $(c_l + ub)^T y \leq 0$, $l = 1, \ldots, p$ (see the proof for theorem 4.1 in Jeyakumar and Li [19]). This further implies Condition 2.3.

In the end of this subsection, we give an example to show that Condition 2.3 strictly dominates the condition in Theorem 2.5. Consider the following GETRS

$$
\begin{aligned}
\min_{x \in \mathbb{R}^2} \quad & x_1^2 - x_2^2 \\
\text{s.t.} \quad & x_1^2 + x_2^2 - 1 \leq 0, \\
& x_1 - 1 \leq 0, \\
& x_2 - 1 \leq 0.
\end{aligned}
$$

One can verify that Condition 2.3 holds, but the dimension condition in Theorem 2.5 fails as dim (Ker$Q$) = 0 but $p = 1$.

2.4. S-Lemma

Using the previous exactness conditions for the GETRS, we further have the following S-lemma with a conic linear inequality.

Theorem 2.6 (S-Lemma with Linear Inequalities). *Suppose Assumptions 2.1, 2.2, and Condition 2.2 hold; and there exists an optimal solution for* (SDP$_0$). *Then the following two statements are equivalent:*

1. $\frac{1}{2}x^T Bx + b^T x + e \leq 0$ *and* $A^T x - d \leq_{\mathcal{K}} 0 \Longrightarrow \frac{1}{2}x^T Cx + c^T x + \gamma \geq 0$.
2. $\exists u \geq 0$, $v \geq_{\mathcal{K}^*} 0$ *such that* $\frac{1}{2}x^T Cx + c^T x + \gamma + u\left(\frac{1}{2}x^T Bx + b^T x + e\right) + v^T (A^T x - d) \geq 0, \quad \forall x \in \mathbb{R}^n$.

Proof. It is obvious that statement 2 implies statement 1. Next let us prove the other direction. From Theorem 2.1, we obtain that the SDP relaxation is bounded from below and $v(\text{SDP}_0) = v(\text{P}_0)$. Then from Assumptions 2.1, we know that strong duality holds between (SDP$_0$) and its dual, which is also the Lagrangian dual of Problem (P$_0$). Hence, we have

$$
\begin{aligned}
& \max_{u \geq 0, v \geq_{\mathcal{K}^*} 0} \min_x L(x, u, v) := \frac{1}{2}x^T Cx + c^T x + u\left(\frac{1}{2}x^T Bx + b^T x + e\right) + v^T (A^T x - d) \\
& = v(\text{SDP}_0) \\
& = v(\text{P}_0) \\
& = \min_x \left\{ \frac{1}{2}x^T Cx + c^T x \,\middle|\, \frac{1}{2}x^T Bx + b^T x + e \leq 0, A^T x - d \leq_{\mathcal{K}} 0 \right\}.
\end{aligned}
$$

Thus,

$$
\min_x \left\{ \frac{1}{2}x^T Cx + c^T x \,\middle|\, \frac{1}{2}x^T Bx + b^T x + e \leq 0, A^T x - d \leq_{\mathcal{K}} 0 \right\} \geq -\gamma
$$

is equivalent to

$$
\max_{u \geq 0, v \geq_{\mathcal{K}^*} 0} \min_x L(x, u, v) := \frac{1}{2}x^T Cx + c^T x + u\left(\frac{1}{2}x^T Bx + b^T x + e\right) + v^T (Ax - d) \geq -\gamma.
$$

This implies that $\exists u \geq 0, v \geq_{\mathcal{K}^*} 0$ such that

$$
\frac{1}{2}x^T Cx + c^T x + \gamma + u\left(\frac{1}{2}x^T Bx + b^T x + e\right) + v^T (Ax - d) \geq 0 \quad \forall x \in \mathbb{R}^n,
$$

which is exactly statement 2. □

Remark 2.4. *The classical S-lemma, which was first proposed by Yakubovich [37], and its variants have a lot of applications in the real world (see the survey paper Pólik and Terlaky [32]). To the best of our knowledge, our S-lemma is the most general one with conic linear inequalities, whereas the S-lemma in Jeyakumar and Li [18] is confined to a unit ball constraint and linear inequalities.*

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2.5. Case Where I_{PSD} Is a Singleton

In this subsection, we discuss the case where I_{PSD} is a singleton. Although in practice this case is numerically unstable, we present a theoretical study here for this case for the sake of completeness.

A natural extension of (P_1) is the following relaxation for (P_0),

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & f_1(z) + \lambda_1 f_2(z) \le t, \\ & A^T z - d \le_{\mathcal{K}} 0. \end{aligned} \tag{11}$$

In general, Problem (11) is not an exact relaxation of (P_0) as the original constraint $f_2(z) \le 0$ may not hold at optimal solutions of (11). Moreover, in this case, the dual problem of (SDP_0) does not satisfy the Slater condition. However, the SOCP based relaxation ($SOCP_{SD}$) (and thus (SDP_0)) may still be exact. Let the notation be the same with Section 2.2. We will modify Condition 2.1 to get a sufficient condition for the exactness of the SDP relaxation of (11) in the case where $I_{PSD} = \{\bar{u}\}$. If I_{PSD} is a singleton, then $J\backslash\hat{J}$ must be nonempty. We still have $\alpha_i \ne 0$ for all $i \in J\backslash\hat{J}$. Indeed, if $\alpha_i = 0$ for some $i \in J\backslash\hat{J}$, then we must have $\delta_i = -\bar{u}\alpha_i = 0$, contradicting $i \in J\backslash\hat{J}$.

Condition 2.4. *The set $I_{PSD} := \{\bar{u}\}$ is a singleton and $J\backslash\hat{J} \ne \emptyset$. Furthermore, one of the following holds:*

1. *If $\alpha_i > 0, \ \ \forall i \in J\backslash\hat{J}$, there exists a $z \in \mathbb{R}^{|J|}$ such that $z_i \ne 0$ for some $i \in J\backslash\hat{J}$, $\bar{A}_J^T z \le_{\mathcal{K}} 0$, and $(\varepsilon_J + \bar{u}\beta_J)^T z \le 0$;*
2. *Else if $\alpha_i < 0, \ \ \forall i \in J\backslash\hat{J}$, there exists a $z \in \mathbb{R}^{|J|}$ such that $z_i \ne 0$ for some $i \in J\backslash\hat{J}$, $\bar{A}_J^T z \le_{\mathcal{K}} 0$, and $(\varepsilon_J + \bar{u}\beta_J)^T z \le 0$;*
3. *Otherwise, there exists two nonzero vectors $z^{(j)} \in \mathbb{R}^{|J|}$ such that $\bar{A}_J^T z^{(j)} \le_{\mathcal{K}} 0$, $(\varepsilon_J + \bar{u}\beta_J)^T z^{(j)} \le 0$, $j = 1, 2$, $\sum_{i\in J}\alpha_i (z_i^{(1)})^2 > 0$, and $\sum_{i\in J}\alpha_i (z_i^{(2)})^2 < 0$.*

We now give some remarks on Condition 2.4. Note that the case where I_{PSD} is a singleton can be seen as a limit case of the case where I_{PSD} is an interval and the two endpoints are approaching to each other. Note also that because of Lemma 2.1, Condition 2.1 implies that α_i have the same signs for $i \in J\backslash\hat{J}$. Hence, cases 1 and 2 in Condition 2.4 coincide with Condition 2.1. Case 3 of Condition 2.2 can be seen as a generalization of Condition 2.1 because it covers all the two possible cases of Condition 2.1. To see this, suppose $I_{PSD} = [\lambda_1, \lambda_2]$ and $J(\lambda_1)\backslash\hat{J} \ne \emptyset$. Then Condition 2.1 implies $\alpha_i > 0$ for $i \in J(\lambda_1)\backslash\hat{J}$ because of Lemma 2.1 and thus $\sum_{i\in J(\lambda_1)} \alpha_i z_i^2 > 0$ because $z_i \ne 0$ for some $i \in J(\lambda_1)\backslash\hat{J}$. With similar arguments, Condition 2.1 also implies $\sum_{i\in J(\lambda_2)} \alpha_i z_i^2 < 0$.

In general, Condition 2.4 is not easy to check. However, when $\alpha_i > 0 \ \ \forall i \in J\backslash\hat{J}$, or $\alpha_i < 0 \ \ \forall i \in J\backslash\hat{J}$, Condition 2.4 reduces to the solvability of a (conic) linear system that can be easily checked. This holds trivially when the cardinality of the index set $J\backslash\hat{J}$ is one.

We show that Condition 2.4 guarantees the exactness of the SDP relaxation of (P_0) in the following theorem.

Theorem 2.7 *Suppose that Assumptions 2.1 and 2.2 hold, and Condition 2.4 holds. Suppose further that there exists an optimal solution for ($SOCP_{SD}$). Then $v(P_0) = v(SOCP_{SD}) = v(SDP_0)$.*

Proof. Because I_{PSD} is a singleton, B must not be p.s.d. Otherwise for $\bar{u} \in I_{PSD}$, we have $\bar{u} + \epsilon \in I_{PSD}, \ \ \forall \epsilon > 0$, which is a contradiction.

Suppose that (x^*, y^*) is an optimal solution of ($SOCP_{SD}$). Then the fulfillment of the Slater condition (Assumptions 2.1) implies that (x^*, y^*) must satisfy the KKT conditions of ($SOCP_{SD}$). Similar to the proof of Theorem 2.1, the complementary slackness $w_i((x_i^*)^2 - y_i^*) = 0$ implies $(x_i^*)^2 = y_i^*$, $i \in J^C$.

Now assume that there exists some $i \in J\backslash\hat{J}$, such that $(x_i^*)^2 < y_i^*$. Let us consider two cases in case 3 of Condition 2.4:

- If $\sum_{i\in J} \alpha_i (y_i^* - (x_i^*)^2) \ge 0$, we consider the following quadratic equation with parameter θ,

$$\sum_{i\in J} \frac{1}{2}\alpha_i (x_i^* + \theta z_i)^2 + \beta_i (x_i^* + \theta z_i) - \left(\sum_{i\in J} \frac{1}{2}\alpha_i y_i^* + \beta_i x_i^*\right) = 0, \tag{12}$$

where z satisfies $\bar{A}_J^T z \le 0$, $(\varepsilon_J + \bar{u}\beta_J)^T z = 0$ and $\sum_{i\in J}\alpha_i z_i^2 > 0$, whose existence is due to Condition 2.4. Note that when $\theta = 0$, the left-hand side (LHS) of (12) becomes $\frac{1}{2}\sum_{i\in J}\alpha_i (x_i^*)^2 - \alpha_i y_i^* \le 0$ as assumed. The LHS of (12) $\to +\infty$ as $\theta \to +\infty$ due to $\sum_{i\in J}\alpha_i z_i^2 > 0$. So we can find a nonnegative solution of (12), denoted by θ^* ($\theta^* = 0$ if

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$\sum_{i\in J}\alpha_i(x_i^*)^2-\alpha_i y_i^*=0$). Now by setting $\bar{x}_{J^C}=x^*_{J^C}$, $\bar{x}_J=x_J+\theta^* z$, and $\bar{y}_i=(\bar{x}_i)^2$ for all $i=1,\dots,n$, we have

$$\sum_{i\in J}\frac{1}{2}\alpha_i\bar{y}_i+\beta_i\bar{x}_i=\sum_{i\in J}\frac{1}{2}\alpha_i y_i^*+\beta_i x_i^*.$$

Hence, because of Condition 2.4, we know that $(\bar{x},\bar{y})$ is a feasible solution to ($\mathrm{SOCP_{SD}}$). Similar to the proof of Theorem 2.1, we also conclude that the objective value of the new solution does not increase from

$$\begin{aligned}
\sum_{i\in J}\frac{1}{2}\delta_i\bar{x}_i^2+\varepsilon_i\bar{x}_i &= \sum_{i\in J}\frac{1}{2}\delta_i(x_i^*+\theta^* z_i)^2+\varepsilon_i(x_i^*+\theta^* z_i)\\
&= \sum_{i\in J}(\varepsilon_i+\bar{u}\beta_i)(x_i^*+\theta^* z_i)-\bar{u}\left[\frac{1}{2}\alpha_i(x_i^*+\theta^* z_i)^2+\beta_i(x_i^*+\theta^* z_i)\right]\\
&= \sum_{i\in J}(\varepsilon_i+\bar{u}\beta_i)(x_i^*+\theta^* z_i)-\bar{u}\left[\frac{1}{2}\alpha_i y_i^*+\beta_i x_i^*\right]\\
&= \sum_{i\in J}\frac{1}{2}\delta_i y_i^*+\varepsilon_i x_i^*+\theta^*(\varepsilon_i+\bar{u}\beta_i)z_i\\
&\le \sum_{i\in J}\frac{1}{2}\delta_i y_i^*+\varepsilon_i x_i^*,
\end{aligned}$$

where the second and fourth equalities are due to $\delta_i+\bar{u}\alpha_i=0,\ \forall i\in J$, the third equality is due to (12), and the inequality is due to Condition 2.4 and $\theta^*\ge 0$. Therefore, $\bar{x}$ is an optimal solution of Problem ($\mathrm{P_0}$) and thus $v(\mathrm{P_0})=v(\mathrm{SOCP_{SD}})$.

• If $\sum_{i\in J}\alpha_i(y_i^*-(x_i^*)^2)<0$, because of the existence of z satisfying that $\bar{A}_J^T z\le 0$, $(\varepsilon_J+\bar{u}\beta_J)^T z=0$, and $\sum_{i\in J}\alpha_i z_i^2<0$, similar arguments yield the exactness of the relaxation ($\mathrm{SOCP_{SD}}$).

Note that when $\alpha_i\ge 0$ for all $i\in J$, we have $J\backslash\hat{J}=\{i:\alpha_i>0, i\in J\}$. Thus, $(x_i^*)^2<y_i^*$ for some $i\in J\backslash\hat{J}$ implies that $\sum_{i\in J}\alpha_i(y_i^*-(x_i^*)^2)>0$. Because of case 1 of Condition 2.4, it follows from $z_i\ne 0$ for some $i\in J\backslash\hat{J}$ that $\sum_{i\in J}\alpha_i z_i^2>0$. Therefore, this case can be proved by the same techniques in the previous proof in the first bullet. Similarly, case 2 of Condition 2.4 can be proved in an analogous way to the second bullet. □

We can also rewrite Condition 2.4 with notation in the original space, which may be more convenient to check.

Condition 2.5. *The two matrices C and B are SD. The set $I_{PSD}:=\{\bar{u}\}$ is a singleton and* $\mathrm{Null}(C+\bar{u}B)\backslash((\mathrm{Null}(C)\cap\mathrm{Null}(B))\ne\{0\}$. *Furthermore, one of the following holds:*

1. *If $s^T Bs>0$ for all $s\in\mathrm{Null}(C+\bar{u}B)\backslash(\mathrm{Null}(C)\cap\mathrm{Null}(B))$, there exists a $z\in\mathbb{R}^n$ such that $z\notin\mathrm{Null}(C)\cap\mathrm{Null}(B)$, $(C+\bar{u}B)z=0$, $A^T z\le_{\mathcal{K}}0$ and $(c+\bar{u}b)^T z\le 0$.*
2. *If $s^T Bs<0$ for all $s\in\mathrm{Null}(C+\bar{u}B)\backslash(\mathrm{Null}(C)\cap\mathrm{Null}(B))$, there exists a $z\in\mathbb{R}^n$ such that $z\notin\mathrm{Null}(C)\cap\mathrm{Null}(B)$, $(C+\bar{u}B)z=0$, $A^T z\le_{\mathcal{K}}0$ and $(c+\bar{u}b)^T z\le 0$.*
3. *Otherwise, there exists two nonzero vectors $z^{(j)}\in\mathbb{R}^n$ such that $A^T z^{(j)}\le_{\mathcal{K}}0$, $(c+\bar{u}b)^T z^{(j)}\le 0$, $j=1,2$, and $(z^{(1)})^T Bz^{(1)}>0$ and $(z^{(2)})^T Bz^{(2)}<0$.*

3. Diagonal QCQPs

In this section, we first compare Condition 2.1 with the recent results in Burer and Ye [9]. Then based on our results in Section 2, we propose sufficient conditions to guarantee the exactness of the SDP relaxation for the following *diagonal* QCQP,

$$\begin{aligned}
\min\quad & \frac{1}{2}x^T Qx+q^T x\\
\text{s.t.}\quad & \frac{1}{2}x^T P_i x+p_i^T x\le r_i\quad \forall i=1,\dots,m,
\end{aligned}\tag{QCQP}$$

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where Q and P_i are all *diagonal* matrices. The SDP relaxation of (QCQP) is

$$\begin{aligned} \min \quad & \frac{1}{2}Q \bullet X + q^T x \\ \text{s.t.} \quad & \frac{1}{2}P_i \bullet X + p_i^T x \le r_i \quad \forall i = 1, \ldots, m, \\ & Y(x, X) := \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0. \end{aligned} \tag{QSDP}$$

Burer and Ye [9] proposed a method to bound the rank of any optimal solution of the SDP relaxation of diagonal QCQPs. Their results further give a sufficient condition for the solution of the SDP relaxation to be of rank one, which means the SDP relaxation is exact. We will next restate their condition and then compare their condition (restricted to the GETRS case) with Condition 2.1. As Q and P_i are all diagonal matrices, (QSDP) is equivalent to the following SOCP problem,

$$\begin{aligned} \min \quad & \frac{1}{2}Q \bullet X + q^T x \\ & \frac{1}{2}P_i \bullet X + p_i^T x \le r_i \quad \forall i = 1, \ldots, m, \\ & X_{ii} \ge x_i^2, \; i = 1, \ldots, n, \end{aligned} \tag{QSOCP}$$

in the sense that if (X^*, x^*) is an optimal solution of (QSDP), then (X^*, x^*) is an optimal solution of (QSOCP);and if $(\bar{X}, \bar{x})$ is an optimal solution of (QSOCP), then $(\tilde{X}, \bar{x})$ is an optimal solution of (QSDP), where $\tilde{X}_{ii} = \bar{X}_{ii}, \; i = 1, \ldots, n$ and $\tilde{X}_{ij} = \bar{x}_i \bar{x}_j, \; i \ne j$.

Now let us first consider the following linear system, which plays an essential role in Burer and Ye [9],

$$\begin{aligned} & \frac{1}{2}Q \bullet X + q_j x_j = -1 \\ & \frac{1}{2}P_i \bullet X + p_{ij} x_j \le 0 \quad \forall i = 1, \ldots, m \\ & X \text{ diagonal}, \quad X_{kk} \ge 0 \quad \forall k \ne j \\ & X_{jj} \text{ free}, x_j \text{ free}, \end{aligned} \tag{13}$$

where p_{ij} denotes the jth entry of the vector p_i. Note that although different from the original system in Burer and Ye [9], there is a coefficient $\frac{1}{2}$ before the terms $Q \bullet X$ and $P_i \bullet X$ in (13), the two systems are essentially the same with a scaling in X. Denote $f := |\{j : (13) \text{ is feasible}\}|$, where f is called the *feasibility number* for (QCQP) Burer and Ye [9]. Now we restate below theorem 1 in Burer and Ye [9].

Theorem 3.1 (Burer and Ye [9, theorem 1]). *Suppose that the feasible region of (QCQP) is nonempty, there exists $\lambda \ge 0$ such that $\sum_{i=1}^m \lambda_i P_i \succ 0$, and the interior feasible region of (QSDP) is nonempty. Let $Y^* := Y(x^*, X^*)$ be any optimal solution of (QSDP). It holds that*

$$1 \le \operatorname{rank}(Y^*) \le n - f + 1.$$

A direct result of Theorem 3.1 is that when $f = n$, the SDP relaxation of (QCQP) is exact. We give a proof in a different view from Burer and Ye [9], which motivates our new sufficient condition in Theorem 3.2.

Proof. The assumption ensures the existence of an optimal solution for (QSDP) and further for (QSOCP) (Burer and Ye [9]). Suppose that (X^*, x^*) is an optimal solution of (QSOCP). We claim that the feasibility of (13) implies that $X_{jj}^* = (x_j^*)^2$. Indeed, letting $(\hat{X}, \hat{x})$ (completing $\hat{x}_k = 0 \;\; \forall k \ne j$) be a feasible solution of (13), if $X_{jj}^* > (x_j^*)^2$, we obtain a new feasible solution $(X^* + \theta\hat{X}, x^* + \theta\hat{x})$ of (QSOCP) with a smaller objective value for all θ satisfying $\theta > 0$ and $X_{jj}^* + \theta\hat{X}_{jj} \ge (x_j^* + \theta\hat{x}_j)^2$. This contradicts the optimality of (X^*, x^*).

Hence, there are at least f-many indices of j such that $X_{jj}^* = (x_j^*)^2$. This implies that $X^* - (x^*)(x^*)^T$ has at least f zeros in diagonal entries. That is, $\operatorname{rank}(X^* - (x^*)(x^*)^T) \le n - f$ and thus $\operatorname{rank}(Y^*) \le n - f + 1$. □

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Now we give a comparison between the results in Burer and Ye [9] and Condition 2.1 for the GETRS with $\mathcal{K} = \mathbb{R}^n_+$. To meet the assumptions in Burer and Ye [9], suppose further that $\mathrm{int}(I_{PSD}) \neq \emptyset$, Assumptions 2.1 and 2.2 hold for the GETRS, and there exists an optimal solution for the SDP relaxation of the GETRS. Note that in Burer and Ye [9], the assumption that there exists $\lambda \geq 0$ with $\sum_{i=1}^m \lambda_i A_i \succ 0$, together with the other two assumptions in Theorem 3.1, ensures that

$$\begin{cases} \text{there exists an optimal solution for (QSDP)}, \\ \text{and strong duality holds between (QSDP) and its dual.} \end{cases} \tag{14}$$

When applying the results in Burer and Ye [9] for the GETRS, we may remove the assumption that there exists $\lambda \geq 0$ such that $\sum_{i=1}^m \lambda_i A_i \succ 0$ because (14) holds under our assumptions. Before comparing the results in Burer and Ye [9] and Condition 2.1, we should point out that Condition 2.2, which is equivalent to Condition 2.1, applies directly to general quadratic forms, whereas Theorem 3.1 only applies to diagonal C and B. Although we can use a change of variables with a congruence matrix as C and B are SD, it costs $O(n^3)$ time for such a simultaneous diagonalization transformation, which is much more expensive than the generalized eigenvalue computation in Condition 2.2.

Now suppose that C and B are both diagonal matrices. Let A_j be the jth column of matrix A. Then system (13) reduces to

$$\begin{aligned}
&\frac{1}{2} C \bullet X + c_j x_j = -1 \\
&\frac{1}{2} B \bullet X + b_j x_j \leq 0 \\
&A_j^T x_j \leq 0 \\
&X \text{ diagonal}, \quad X_{kk} \geq 0 \quad \forall k \neq j \\
&X_{jj} \text{ free}, x_j \text{ free}.
\end{aligned} \tag{15}$$

Recall $J(u) = \{i : \delta_i + u\alpha_i = 0,\ i = 1, \ldots, n\}$, $u \in \partial I_{PSD}$, and $\hat{J} = \{i : \delta_i = \alpha_i = 0,\ i \in J\}$. Set $L(u) = J(u) \backslash \hat{J}$. For simplicity, we will use J and L instead of $J(u)$ and $L(u)$, respectively, if it does not cause any confusion. Then Condition 2.1 is equivalent to the solvability of the following system

$$\begin{aligned}
&(c_J + ub_J)^T x \leq 0 \\
&A_J^T x \leq 0 \\
&x_i \neq 0 \ \text{ for some } i \in L
\end{aligned} \tag{16}$$

for all $u \in \partial I_{PSD}$. We observe the following differences between (15) and (16) (i.e., the conditions in Burer and Ye [9] and our paper) in guaranteeing the exactness of the SDP relaxation of the GETRS:

1. The techniques are different. In fact, the original proof associated with (15) in Burer and Ye [9] uses the complementary slackness of primal and dual solutions of the SDP relaxation to guarantee the exactness of the SDP relaxation, though in Theorem 3.1 we also give a proof in a view of feasible directions. On the other hand, the proofs in Theorems 2.1 and 2.2 associated with (16) use an idea of feasible directions. The conditions in Burer and Ye [9] force every optimal solution to be of rank one. Our conditions guarantee that there exists an optimal solution with rank one but do not require all optimal solutions to be of rank one. One reason is that the first inequality in (15) is a strict inequality $\frac{1}{2} C \bullet X + c_j x_j < 0$ (we rewrite $= -1$ to <0 because of the homogeneousness of the system), whereas the first inequality in (16) may achieve zero. In the view of feasible directions, we also immediately see why Conditions 2.1 and 2.2 do not depend on the constants in the quadratic and linear constraints.

2. We note that the conditions in Theorem 3.1 with $f = n$ need to solve n different linear systems in forms of (15), where the dimension of each system is $n + 1$, whereas (16) involves at most two much smaller dimensional linear systems (with dimension $|J|$) with respect to $u = \lambda_1$ or λ_2.

3. We note that (15) is a coordinate-wise condition, whereas (16) can handle several coordinates together as J may contain more than one index. If J contains only one index, then (15) is still incomparable with (16). Particularly, (15) involves the lifted variable X, whereas (16) does not because $C_{ii} + uB_{ii} = 0$ for each $i \in J \backslash \hat{J}$.

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4. The results in Burer and Ye [9], to the best of our knowledge, do not apply to the conic linear constraint $A^Tz - d \leq_{\mathcal{K}} 0$, whereas ours do. Moreover, our results also give convex relaxations (P_1) and (P_2) without using a simultaneous diagonalization transformation.

Example 3.1. *To further compare our Condition (16) and Burer and Ye's results in Theorem 3.1, we consider the following example:*

$$\begin{aligned} \min \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & -x_1^2 + x_2^2 \leq 1. \end{aligned}$$

It is easy to verify that $I_{PSD} = [0,1]$ and our Condition (16) holds. So the SDP relaxation is exact. On the other hand, for $j = 1$, system (15) becomes

$$\begin{aligned} & X_{11} + X_{22} = -1 \\ & -X_{11} + X_{22} \leq 0 \\ & X \textit{ diagonal}, \quad X_{22} \geq 0 \\ & X_{11} \textit{ free}. \end{aligned}$$

Aggregation of the first two constraint becomes $2X_{22} \leq -1$, which is impossible because $X_{22} \geq 0$. If $j = 2$, the system is feasible with $(X_{11}, X_{22}, x_2) = (0, -1, 0)$. So the feasibility number is 1. Then Theorem 3.1 tells us that $1 \leq \text{rank}(Y^*) \leq 2 - 1 + 1 = 2$, which cannot imply the exactness of the SDP relaxation.

In the following, motivated by the idea of constructing a new solution from a special feasible direction in Condition 2.1, we propose a new condition to guarantee the rank of an optimal solution of (QSDP). Now consider the following linear system

$$\frac{1}{2}Q_{jj}X_{jj} + q_jx_j \leq 0, \ \frac{1}{2}(P_i)_{jj}X_{jj} + p_{ij}x_j \leq 0 \ \ \forall i = 1, \ldots, m, \ \text{ either } X_{jj} < 0 \ \text{ or } x_j \neq 0. \tag{17}$$

We then define an alternative *feasibility number* $f' = |\{j : \text{either (13) or (17) is feasible}\}|$.

Theorem 3.2. *Suppose that the feasible region of (QCQP) is nonempty, there exists $\lambda \geq 0$ such that $Q + \sum_{i=1}^m \lambda_i P_i \succ 0$, and the interior feasible region of (QSDP) is nonempty. Let $Y^* := Y(x^*, X^*)$ be an optimal solution of (QSDP). It holds that*

$$1 \leq \text{rank}(Y^*) \leq n - f' + 1.$$

When $f' = n$, the SDP relaxation of (QCQP) is exact.

Proof. The proof is similar to that of Theorem 3.1 with the following additional observations. Suppose that (X^*, x^*) is an optimal solution to (QSOCP) with $(x_j^*)^2 < X_{jj}^*$ for index j. Assume $(\hat{X}_{jj}, \hat{x}_j)$ is a feasible solution of (17). Complete $(\hat{X}, \hat{x})$ by setting all entries except $(\hat{X}_{jj}, \hat{x}_j)$ to be zero. We claim that there exists a $\theta \geq 0$ such that $(\bar{X}, \bar{x}) = (X^* + \theta\hat{X}, x^* + \theta\hat{x})$ is an optimal solution of (QSOCP) with $\bar{X}_{jj} = \bar{x}_j^2$. Indeed, because (i) either $\hat{X}_{jj} < 0$ or $\hat{x}_j^2 > 0$ (due to $\hat{x}_j \neq 0$), (ii) $(x_j^*)^2 - X_{jj}^* < 0$, (iii) $\hat{X}_{st} = 0$ for $(s,t) \neq (j,j)$, and (iv) $\hat{x}_k = 0 \ \ \forall k \neq j$, we obtain that there exists a positive θ such that $(x_j^* + \theta\hat{x}_j)^2 = X_{jj}^* + \theta\hat{X}_{jj}$, and $(\bar{X}, \bar{x}) = (X^* + \theta\hat{X}, x^* + \theta\hat{x})$ is a feasible solution of (QSOCP) that has an objective value no larger than that of (X^*, x^*), which is thus optimal. Note that the only difference between (X^*, x^*) and $(\bar{X}, \bar{x})$ is the jjth entries in $\bar{X}$ and X^* and the jth entries in $\bar{x}$ and x^*. Hence, we conclude that there exists an optimal solution $(\tilde{X}, \tilde{x})$ satisfying $\tilde{X}_{jj} = \tilde{x}_j^2$ for any index j such that system (17) is feasible. □

The condition in Theorem 3.1 makes every optimal solution satisfying $X_{jj} = x_j^2$ for the rank guarantee, whereas our condition in Theorem 3.2 is less restrictive as it only requires the existence of one optimal solution such that $X_{jj} = x_j^2$ if system (17) is feasible for index j. We should point out that one disadvantage of (17) is that the system involves $m + 1$ linear inequalities and only two variables, which may not be easy to be feasible if m is large.

At the end of this section, let us consider again Example 3.1. We can check our alternative feasibility number $f' = 2$ for the same example. Indeed, the system (17) with $j = 1$ reduces to

$$X_{11} \leq 0, \ -X_{11} \leq 0, \ \text{ either } X_{11} < 0 \text{ or } x_1 \neq 0,$$

where $(X_{11}, x_1) = (0, 1)$ is a feasible solution. Hence, together with the fact that the system (13) is feasible for $j = 2$ (as indicated in Example 3.1), we conclude $f' = 2$ and it follows that the rank of the SDP solution is one from Theorem 3.2.

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4. Conclusions

In this paper, we investigate sufficient conditions to guarantee the exactness of the SDP relaxation for the GETRS. We propose two different classes of convex relaxations of the GETRS, which are equivalent to the basic SDP relaxation. We also propose sufficient conditions to guarantee the exactness of the two classes of the convex relaxations under mild assumptions in different cases. The two convex relaxations also offer alterative efficient methods to solve the GETRS instead of the SDP relaxation. Based on our sufficient conditions, we also obtain a more general S-lemma than that in Jeyakumar and Li [18]. Finally, we compare our results with a recent sufficient condition in Burer and Ye [9] and also give a new sufficient condition to bound the rank of solutions of the SDP relaxation for diagonal QCQPs.

When the quadratic constraint becomes an equality in the GETRS, our previous sufficient conditions still guarantee, albeit with a slight modification, the exactness of the SDP relaxation. Because the technique is similar, we omit the details for this variant. We believe that our work in this paper could highlight a connection between general QCQPs and the GETRS, one simplest class of QCQPs, and shed some light on the exactness of SDP relaxations for general QCQPs. For future research directions, we will investigate more general sufficient conditions to guarantee the exactness of the SDP relaxation and extend our sufficient conditions in this paper to general QCQPs. We would also like to find more real-world applications of our newly developed S-lemma.

Acknowledgments

The authors thank the associate editor and two anonymous referees for their invaluable comments that improved the quality of the paper significantly and especially for bringing the reference Burer and Ye [9] to the authors' attention.

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