Online Supplement to Complexity Results and Effective Algorithms for Worst-case Linear Optimization under Uncertainties

Hezhi Luo* Xiaodong Ding† Jiming Peng‡ Rujun Jiang§ Duan Li¶

August 25, 2019

1 Proofs of Propositions, Lemmas and Theorems

In this section, we provide the proofs of theorems, lemmas and propositions in the main body of the paper, and gives additional supporting results needed for these proofs.

To prove Proposition 1, we first give a technical lemma as follows.

Lemma A.1 *Let $Q^T y \neq 0$ and $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\|Q^T y\|_q = \max_{u \in \mathbb{R}^r} \left\{ u^T Q^T y : \|u\|_p \leq 1 \right\}. \tag{41}$$

Proof. Let $Q^T y \neq 0$. By the Hölder inequality, we have

$$u^T Q^T y \leq \|u\|_p \|Q^T y\|_q,$$

where $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and the equality holds if and only if $|u_i|^p = \lambda |q_i^T y|^q$, $i = 1, \ldots, r$ for some $\lambda > 0$ and $\arg(u, Q^T y) = 0$. Thus, problem (41) attains its maximum if and only if $|u_i|^p = \lambda |q_i^T y|^q$, $i = 1, \ldots, r$ for some $\lambda > 0$, $\arg(u, Q^T y) = 0$ and $\|u\|_p = 1$. The later yields $\lambda = \|Q^T y\|_q^{-\frac{q}{p}}$ and $u \in \mathbb{R}^r$ is given by

$$u_i = \text{sign}\left(q_i^T y\right) \left|q_i^T y\right|^{\frac{q}{p}} \left\|Q^T y\right\|_q^{-\frac{q}{p}}, \quad i = 1, \ldots, r,$$

*Department of Mathematics, College of Science, Zhejiang Sci-Tech University, Hangzhou, Zhejiang 310018, P. R. China. Email: hzluo@zstu.edu.cn.

†Department of Applied Mathematics, College of Science, Zhejiang University of Technology, Hangzhou, Zhejiang 310032, P. R. China. Email: dxdopt@zjut.edu.cn.

‡Department of Industrial Engineering, University of Houston, Houston, TX, 77204. Email: jopeng@uh.edu.

§School of Data Science, Fudan University, Shanghai, P. R. China. Email: rjjiang@fudan.edu.cn

¶Corresponding author. School of Data Science, City University of Hong Kong, Hong Kong. Email: dli226@cityu.edu.hk

which, by $\frac{q}{p} = q-1$, implies $u_i = \mu_i(y;q)$, $i = 1,\dots,r$, where $\mu_i(y;q)$ is defined by (1). So, $\mu(y;q)$ is the optimal solution of problem (41), with the optimal value $\|Q^T y\|_q$. □

Proof of Proposition 1 Let z_p^* be the optimal value of (WCLO$_p$) with $p \in (1,\infty)$. Using the strong duality theory for LO, we have

$$\begin{aligned} z_p^* &= \max_{\|u\|_p \le 1} \max_{y\in\mathcal{C}} \left\{ u^T Q^T y + b_0^T y \right\} \\ &= \max_{y\in\mathcal{C}} \left\{ b_0^T y + \max_{u} \left\{ u^T Q^T y : \|u\|_p \le 1 \right\} \right\} \\ &= \max_{y\in\mathcal{C}} \left\{ b_0^T y + \|Q^T y\|_q \right\}, \end{aligned}$$

where $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and the last equality follows from Lemma A.1.

Now, let y^* be a globally optimal solution of problem (2) satisfying $Q^T y^* \neq 0$. Since all the constraints in problem (2) are linear, based on the first order KKT necessary optimality condition, there exist $\bar{x} \in \mathbb{R}_+^n$ and $\bar{v} \in \mathbb{R}_+^m$ such that

$$\begin{cases} -b_0 - Q\mu(y^*;q) + A\bar{x} + \bar{v} = 0, \\ \bar{x}^T (A^T y^* - c) = 0, \quad \bar{v}^T y^* = 0, \end{cases} \tag{42}$$

where $Q\mu(y^*;q)$ is the gradient of $\|Q^T y\|_q$ at point y^*, and $\mu(y^*;q) \in \mathbb{R}^r$ is given by (1). Since $\frac{1}{p} + \frac{1}{q} = 1$, via simple calculus we obtain from (1) that

$$\|\mu(y^*;q)\|_p = 1, \qquad \mu(y^*;q)^T Q^T y^* = \|Q^T y^*\|_q. \tag{43}$$

Let $b^* = Q\mu(y^*;q) + b_0$. It then follows from (43) that $b^* \in \mathcal{U}_p$ and $f_q(y^*) = (b^*)^T y^*$. Moreover, we have from (42) that

$$-b^* + A\bar{x} + \bar{v} = 0, \quad \bar{x}^T (A^T y^* - c) = 0, \quad \bar{v}^T y^* = 0, \tag{44}$$

which implies that $(b^*)^T y^* = c^T \bar{x}$, and y^* and $\bar{x}$ are the optimal solutions of problems $\max\limits_{y\in\mathcal{C}} (b^*)^T y$ and $\min\limits_{x\in\mathcal{X}(b^*)} c^T x$, respectively.

On the other hand, since y^* is the globally optimal solution of problem (2) and by the Hölder inequality, we can conclude that, for every $y \in \mathcal{C}$ and $u \in \mathbb{R}^r$ with $\|u\|_p \le 1$, one has

$$f_q(y^*) \ge f_q(y) \ge \|u\|_p \|Q^T y\|_q + b_0^T y \ge u^T Q^T y + b_0^T y.$$

This, together with $f_q(y^*) = (b^*)^T y^*$ and the definition of $\mathcal{U}_p$, implies that

$$b^T y \le (b^*)^T y^*, \quad \forall y \in \mathcal{C}, \quad \forall b \in \mathcal{U}_p. \tag{45}$$

For a given $b \in \mathcal{U}_p$, let us define

$$\psi(b) = \min_{x\in\mathcal{X}(b)} c^T x. \tag{46}$$

From the strong duality theory of LO and by (45), we have

$$\psi(b) = \max_{y\in\mathcal{C}} b^T y \le (b^*)^T y^*, \quad \forall b \in \mathcal{U}_p. \tag{47}$$

Since $b^* \in \mathcal{U}_p$ and $y^* \in \mathcal{C}$, from (47) we have $\psi(b^*) \le (b^*)^T y^* \le \psi(b^*)$. It then follows that $\psi(b^*) = (b^*)^T y^*$. This, together with (47), implies that $\psi(b) \le \psi(b^*)$, $\forall b \in \mathcal{U}_p$. Therefore, b^* is a globally optimal solution of problem $\max_{b\in\mathcal{U}_p} \psi(b)$, which further implies that b^* is a globally optimal solution of (WCLO$_p$). □

Proof of Proposition 4 Let z^*_∞ be the optimal value of (WCLO$_\infty$). By the strong duality theory of LO, we have

$$\begin{aligned}
z^*_\infty &= \max_{\|u\|_\infty\le 1} \max_{y\in\mathcal{C}} \left\{u^T Q^T y + b_0^T y\right\} \\
&= \max_{y\in\mathcal{C}} \left\{b_0^T y + \max_{\|u\|_\infty\le 1} u^T Q^T y\right\} \\
&= \max_{y\in\mathcal{C}} \left\{b_0^T y + \sum_{i=1}^m \mathrm{sign}(q_i^T y) q_i^T y\right\} \\
&= \max_{y\in\mathcal{C}} \left\{\|Q^T y\|_1 + b_0^T y\right\},
\end{aligned}$$

where $\mathrm{sign}(w) = 1$ if $w \ge 0$ and -1 else.

Let y^* be the globally optimal solution of problem (3) and u^* be defined by (4). Using the Hölder inequality, we derive the following relation

$$u^T Q^T y + b_0^T y \le \|u\|_\infty \|Q^T y\|_1 + b_0^T y \le f_1(y) \le f_1(y^*),\ \forall y \in \mathcal{C}, \forall u : \|u\|_\infty \le 1.$$

Let $b^* = Qu^* + b_0$. By (4), it is easy to verify $f_1(y^*) = (b^*)^T y^*$. Therefore,

$$b^T y \le (b^*)^T y^*,\ \forall y \in \mathcal{C}, \forall b \in \mathcal{U}_\infty. \tag{48}$$

Using the same arguments as in the proof of Proposition 1, we can infer from (48) that b^* is a globally optimal solution of (WCLO$_\infty$). □

Proof of Proposition 6 By the strong duality theory of LO, (WCLO$_1$) can be reformulated as

$$z_1^* = \max\left\{u^T Q^T y + b_0^T y : y \in \mathcal{C},\ \|u\|_1 \le 1\right\}. \tag{49}$$

For fixed $y \in \mathcal{C}$, problem (49) reduces to the following LO problem,

$$\max\ (Q^T y)^T u \quad \text{s. t.} \quad \|u\|_1 \le 1, \tag{50}$$

whose maximum can be attained at u^* where

$$u_i^* = \begin{cases} \text{sign}(q_i^T y) & \text{if } |q_i^T y| = \|Q^T y\|_\infty; \\ 0 & \text{otherwise.} \end{cases} \tag{51}$$

In other words, problem (50) attains its maximum when $u^* = e_i$ for some suitable $i \in \{1, \dots, 2r\}$. Thus, problem (49) can be reformulated as the following,

$$\begin{aligned} z_1^* &= \max_{y \in \mathcal{C}} \max_{i=1,\dots,2r} \left\{ e_i^T Q^T y + b_0^T y \right\} \\ &= \max_{i=1,\dots,2r} \max_{y \in \mathcal{C}} \left\{ e_i^T Q^T y + b_0^T y \right\} \\ &= \max_{i=1,\dots,2r} \min \left\{ c^T x : Ax \le Qe_i + b_0,\ x \ge 0 \right\} \\ &= \max_{i=1,\dots,2r} \{ c^T \bar{x}^i \}, \end{aligned} \tag{52}$$

where the third equality follows from the strong duality of LO and $\bar{x}^i$ is the optimal solution of problem (5) with index i.

Recall that for any given $b \in \mathcal{U}_1$, $\psi(b)$ is defined by (46). From the strong duality theory of LO we obtain

$$\psi(b) = \max_{y \in \mathcal{C}} b^T y, \quad \forall b \in \mathcal{U}_1, \tag{53}$$

Let $i_0 = \arg \max_{i=1,\dots,2r} \{c^T \bar{x}^i\}$ and $b^* = Qe^{i_0} + b_0$. Clearly, $b^* \in \mathcal{U}_1$. From (52), we have

$$z_1^* = c^T \bar{x}^{i_0} = \max_{y \in \mathcal{C}} (b^*)^T y = \psi(b^*). \tag{54}$$

The optimality of z_1^* in (49) further yields that $\psi(b) = \max_{y \in \mathcal{C}} b^T y \le z_1^*$, $\forall b \in \mathcal{U}_1$, which, by (54), in turn implies that $\psi(b) \le \psi(b^*)$ for any $b \in \mathcal{U}_1$. This shows that b^* is the optimal solution of (WCLO_1). □

Proof of Theorem 1 The proof is similar to that of Theorem 1 in [2]. For self-completeness, we give the detailed proof here. We show that problem (2) with $q \in (1, \infty)$ reduces to the strongly NP-hard 3-partition problem in [1]. Given a set S with $n = 3m$ integers $\{a_1, a_2, \dots, a_n\}$, the sum of S is equal to $m\rho$ and each integer in S is strictly between $\rho/4$ and $\rho/2$. The 3-partition problem is to decide whether S can be partitioned into m subsets such that the sum of the numbers in each subset is equal to each other, i.e., ρ, which implies each subset has exactly three elements.

Let $S = \{a_1, a_2, \dots, a_n\}$ with $n = 3m$, where $\sum_{i=1}^n a_i = m\rho$ and each $a_i \in (\rho/4, \rho/2)$.

Consider the following maximization problem:

$$\begin{aligned} \max \quad & \hat{\ell}(x) := \sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij}^q \\ \text{s.t.} \quad & \sum_{j=1}^{m} x_{ij} = 1,\ i = 1,\dots,n, \\ & \sum_{i=1}^{n} a_i x_{ij} = \rho,\ j = 1,\dots,m, \\ & x_{ij} \geq 0,\ i = 1,\dots,n,\ j = 1,\dots,m. \end{aligned} \tag{55}$$

Let x be a feasible solution of problem (55). Since $0 \leq x_{ij} \leq 1$ and $q > 1$, we have $x_{ij}^q \leq x_{ij}$ and thus,

$$\sum_{j=1}^{m} x_{ij}^q \leq \sum_{j=1}^{m} x_{ij} = 1, \quad i = 1,\dots,n,$$

where the equality holds if and only if $x_{ij_0} = 1$ for some j_0 and other $x_{ij} = 0$, $\forall j \neq j_0$. Thus, $\hat{\ell}(x) \leq n$ for any feasible solution x of problem (55).

If there is a feasible solution x such that $\hat{\ell}(x) = n$, then $\sum_{j=1}^{m} x_{ij}^q = \sum_{j=1}^{m} x_{ij} = 1$ for all i so that for any i, $x_{ij_i} = 1$ for some j_i and other $x_{ij} = 0$, $\forall j \neq j_i$. This generates an equitable 3-partition of the entries of S. On the other hand, if the entries of S have an equitable 3-partition, then (55) must have a binary solution x such that $\hat{\ell}(x) = n$. Thus we prove the strong NP-hardness of problem (2) with $q \in (1, \infty)$ according to [1]. $\square$

Proof of Lemma 1 Take $\hat{t} = b_0^T y^0$. Since $\mu = Q^T y^0 \neq 0$, we have

$$g_\mu(\hat{t}, y^0) = (\hat{t} - b_0^T y^0)^2 - 2\mu^T Q^T y^0 + \|\mu\|_2^2 = -\|\mu\|_2^2 < 0,$$

which, by $y^0 \in \mathcal{C}$, implies that both $\mathcal{F}_\mu$ and $\text{int}\mathcal{F}_\mu$ are nonempty. It is easy to see that $\mathcal{F}_\mu$ is a closed convex set. In addition, $g_\mu(t, y) \geq h(t, y)$ implies $\mathcal{F}_\mu \subseteq \mathcal{F}$. $\square$

Proof of Lemma 2 Since $t^0 = b_0^T y^0$ and $y^0 \in \mathcal{C}$, we have $(t^0, y^0) \in \mathcal{F}$. Since $\mu^0 = Q^T y^0 \neq 0$, we obtain $(t^0, y^0) \in \mathcal{F}_{\mu^0}$ and so $\mathcal{F}_{\mu^0} \neq \emptyset$. Step 1 implies that $(t^1, y^1) \in \mathcal{F}_{\mu^0}$. It then follows from Lemma 1 that $(t^1, y^1) \in \mathcal{F}$. By induction, we conclude that $\{(t^k, y^k)\} \subseteq \mathcal{F}$. $\square$

Proof of Lemma 3 From Step 0, $\mu^0 \neq 0$. Suppose that $\mu^{l-1} \neq 0$, $l \geq 1$. Recall from Step 1 that $(t^l, y^l) \in \mathcal{F}_{\mu^{l-1}}$ and so $2(\mu^{l-1})^T Q^T y^l - \|\mu^{l-1}\|_2^2 \geq 0$. This, together with the assumption that $\mu^{l-1} \neq 0$, implies $Q^T y^l \neq 0$. By $\mu^l = Q^T y^l$, we have $\mu^l \neq 0$. We conclude, by induction, that $\mu^k \neq 0$ for all k. On the other hand, by Lemma 1, $\mu^k = Q^T y^k \neq 0$ for all k implies that $\text{int}\mathcal{F}_{\mu^k} \neq \emptyset$ for all k. $\square$

Proof of Lemma 4 By Lemma 2, $\{(t^k, y^k)\} \subseteq \mathcal{F}$. Since $\mu^k = Q^T y^k$ for all k, it is easy to see that

$$g_{\mu^k}(t^k, y^k) = (t^k - b_0^T y^k)^2 - 2(\mu^k)^T Q^T y^k + \|\mu^k\|_2^2 = (t^k - b_0^T y^k)^2 - \|Q^T y^k\|_2^2 \leq 0,$$

where the last inequality is due to $\{(t^k, y^k)\} \subseteq \mathcal{F}$. Hence $(t^k, y^k) \in \mathcal{F}_{\mu^k}$ for all k, and thus (t^k, y^k) is a feasible solution to problem (12) with $\mu = \mu^k$. From Step 1 in the algorithm, since (t^{k+1}, y^{k+1}) is an optimal solution to problem (12) with $\mu = \mu^k$, we deduce $t^k \leq t^{k+1}$ for all k. This proves that $\{t^k\}$ is a nondecreasing sequence. On the other hand, by Lemma 3, $\mu^k \neq 0$ for all k. Note that since the optimal value of problem (12) is a lower bound to problem (8), we have $t^k \leq t^*$ for all k, where t^* is the optimal value of problem (8). Therefore, the sequence $\{t^k\}$ converges. □

Proof of Lemma 5 Let $\mathcal{K}$ be the index set of the subsequence satisfying $\{(t^k, \mu^k, y^k)\}_{\mathcal{K}} \to (\hat{t}, \hat{\mu}, \hat{y})$. By Lemma 2, $\{(t^k, y^k)\} \subseteq \mathcal{F}$. The closedness of $\mathcal{F}$ implies $(\hat{t}, \hat{y}) \in \mathcal{F}$ and thus $\hat{y} \in \mathcal{C}$. Note from Step 1 that, since $\mu^k = Q^T y^k$ for all k, we have $\hat{\mu} = Q^T \hat{y}$. We then follow $(\hat{t}, \hat{y}) \in \mathcal{F}_{\hat{\mu}}$.

Now we prove that $t \leq \hat{t}$ for any $(t, y) \in \mathcal{F}_{\hat{\mu}}$. Lemma 4, $\{t^k\}$ is a nondecreasing and convergent sequence. Take any $(t, y) \in \mathcal{F}_{\hat{\mu}}$. Suppose that $(t, y) \in \mathcal{F}_{\mu^k}$ for some μ^k generated by Algorithm 1. Recall from Step 1 that (t^{k+1}, y^{k+1}) is the optimal solution of problem (12) with $\mu = \mu^k$. We then obtain $t \leq t^{k+1}$. Because of the monotonicity of the sequence $\{t^k\}$ and by the fact that $\hat{t}$ is an accumulation point of $\{t^k\}$, we can infer $t \leq \hat{t}$.

It remains to consider the case where $(t, y) \in \mathcal{F}_{\hat{\mu}} \backslash \mathcal{F}_{\mu^k}$ for all k. Since $\hat{\mu} = Q^T \hat{y} \neq 0$, by Lemma 1, $\mathcal{F}_{\hat{\mu}} \neq \emptyset$ and $\text{int}\mathcal{F}_{\hat{\mu}} \neq \emptyset$. From the proof of Lemma 1, we see that $g_{\hat{\mu}}(\tilde{t}, \hat{y}) < 0$, where $\tilde{t} = b_0^T \hat{y}$. Let $\delta = -g_{\hat{\mu}}(\tilde{t}, \hat{y}) > 0$. Since $g_{\mu}(\tilde{t}, \hat{y})$ is continuous in μ and $\{\mu^k\}_{\mathcal{K}} \to \hat{\mu}$, we have that $g_{\mu^k}(\tilde{t}, \hat{y}) \leq -\delta/2 < 0$ for sufficiently large $k \in \mathcal{K}$. For any given $(t, y) \in \mathcal{F}_{\hat{\mu}} \setminus \mathcal{F}_{\mu^k}$, let us define $\rho_k = \max\left\{0, g_{\mu^k}(t, y)\right\}$. Clearly, $\rho_k > 0$. Since $\{\mu^k\}_{\mathcal{K}} \to \hat{\mu}$ and $(t, y) \in \mathcal{F}_{\hat{\mu}}$, it is easy to verify

$$\lim_{k \in \mathcal{K} \to \infty} \rho_k = 0. \tag{56}$$

Define

$$\lambda_k = 2\rho_k/(2\rho_k + \delta). \tag{57}$$

We thus have

$$0 < \lambda_k < 1, \quad \lim_{k \in \mathcal{K} \to \infty} \lambda_k = 0. \tag{58}$$

Let us define

$$(\hat{t}^k, \hat{y}^k) = (1 - \lambda_k)(t, y) + \lambda_k(\tilde{t}, \hat{y}).$$

Since $g_{\mu^k}(t,y)$ is a convex function in (t,y), by (57), we have

$$g_{\mu^k}(\hat{t}^k,\hat{y}^k) \le (1-\lambda_k)g_{\mu^k}(t,y) + \lambda_k g_{\mu^k}(\tilde{t},\hat{y}) \le (1-\lambda_k)\rho_k + \lambda_k(-\delta/2) = 0,$$

which in turn implies that $(\hat{t}^k,\hat{y}^k) \in \mathcal{F}_{\mu^k}$. Since (t^{k+1},y^{k+1}) is the optimal solution of problem (12) with $\mu=\mu^k$, we have $\hat{t}^k \le t^{k+1}$ and hence

$$\hat{t}^k \le \hat{t}, \tag{59}$$

because of $t^k \le \hat{t}$ for all k. Furthermore, we have

$$\operatorname{dist}\big((t,y),\mathcal{F}_{\mu^k}\big) := \min\big\{\|(\tilde{t},\tilde{y})-(t,y)\| : (\tilde{t},\tilde{y}) \in \mathcal{F}_{\mu^k}\big\} \le \|(\hat{t}^k,\hat{y}^k)-(t,y)\|.$$

Note that $(\hat{t}^k,\hat{y}^k)-(t,y) = \lambda_k[(\tilde{t},\hat{y})-(t,y)]$. Note also that $\mathcal{F}_{\mu^k}$ is a nonempty closed convex set. Let $(\tilde{t}^k,\tilde{y}^k) \in \mathcal{F}_{\mu^k}$ be the projection of (t,y) onto $\mathcal{F}_{\mu^k}$, it follows immediately

$$\|(\tilde{t}^k,\tilde{y}^k)-(t,y)\| = \operatorname{dist}\big((t,y),\mathcal{F}_{\mu^k}\big) \le \|(\hat{t}^k,\hat{y}^k)-(t,y)\| = \lambda_k\|(\tilde{t},\hat{y})-(t,y)\|. \tag{60}$$

From the above relation and (58), we obtain

$$\lim_{k\in\mathcal{K}\to\infty}|\tilde{t}^k - t| = \lim_{k\in\mathcal{K}\to\infty}|\hat{t}^k - t| = 0,$$

which, together with relation (59), further yields the conclusion of the lemma. □

Proof of Proposition 9 Denote $\mathcal{D} = \big\{z \in \mathbb{R}^r : z = Q^T y,\ y \in \mathcal{C}\big\}$. Assume that $J=\emptyset$. Then

$$0 = \max_{j=1,\ldots,\rho}\max_{z\in\mathcal{D}} \xi_j^T z = \max_{z\in\mathcal{D}}\max_{j=1,\ldots,\rho}\big\{\xi_j^T z\big\}. \tag{61}$$

We consider the following two cases:

Case (a): $\rho = 2r$, $\xi_j = e_j$, $\xi_{j+r} = -e_j$, $j=1,\ldots,r$. In this case, from (61), we have

$$0 = \max_{z\in\mathcal{D}}\max_{i=1,\ldots,r}\{|z_i|\} = \max_{z\in\mathcal{D}}\|z\|_\infty \quad\Rightarrow\quad Q^T y = 0, \quad \forall y \in \mathcal{C},$$

which contradicts Assumption 2.

Case (b): $\rho = 2^r$, $\xi_j \in \{-1,1\}^r$ for $j=1,\ldots,\rho$. In this case, for a given $z \in \mathcal{D}$, we choose a vector $\xi_k \in \mathbb{R}^r$ whose i-th component is $\xi_{ki} = 1$ if $z_i \ge 0$ and $\xi_{ki} = -1$ else. It is clear that $\xi_k \in \{-1,1\}^r$ and $\xi_k^T z = \sum_{i=1}^r |z_i|$. Then, for a given $z \in \mathcal{D}$, we have

$$\max_{j=1,\ldots,2^r}\big\{\xi_j^T z\big\} = \xi_k^T z = \sum_{i=1}^r |z_i| = \|z\|_1. \tag{62}$$

It then follows from (61) and (62) that

$$0 = \max_{z\in\mathcal{D}}\|z\|_1 \quad\Rightarrow\quad Q^T y = 0, \quad \forall y \in \mathcal{C},$$

which also contradicts Assumption 2. □

Proof of Theorem 3 Note that the problems (14) and (17) are equivalent in the sense that they have the same optimal solutions and optimal value. Since $(\bar{t}, \bar{y}, \bar{z}, \bar{s})$ is an optimal solution to problem (15), $\bar{y}$ must be a feasible solution to problem (17). Thus, we have $f^*_{[l,u]} \geq f_2(\bar{y})$. It then follows from the choice of $f^*_{[l,u]}$ that

$$f_2(\bar{y}) \leq f^*_{[l,u]} \leq v^*_{[l,u]}. \tag{63}$$

On the other hand, from the constraints in (15), we have

$$\begin{aligned}(\bar{t} - b_0^T \bar{y})^2 - \|Q^T \bar{y}\|_2^2 &= (\bar{t} - b_0^T \bar{y})^2 - \sum_{i=1}^r \bar{s}_i - \|\bar{z}\|^2 + \sum_{i=1}^r \bar{s}_i \\ &\leq \sum_{i=1}^r (\bar{s}_i - \bar{z}_i^2) \\ &\leq \sum_{i=1}^r [-\bar{z}_i^2 + (l_i + u_i)\bar{z}_i - l_i u_i] \\ &\leq \frac{1}{4}\|u - l\|_2^2.\end{aligned} \tag{64}$$

Let $\bar{\delta} = \sum_{i=1}^r [\bar{s}_i - \bar{z}_i^2]$. Clealy, $0 \leq \bar{\delta} \leq \frac{1}{4}\|u - l\|_2^2$. From (64), we have

$$\bar{t} \leq b_0^T \bar{y} + \sqrt{\|Q^T \bar{y}\|_2^2 + \bar{\delta}}.$$

Recall that $v^*_{[l,u]} = \bar{t}$ and $f_2(\bar{y}) = b_0^T \bar{y} + \|Q^T \bar{y}\|_2$. We then have

$$v^*_{[l,u]} - f_2(\bar{y}) \leq \sqrt{\|Q^T \bar{y}\|_2^2 + \bar{\delta}} - \sqrt{\|Q^T \bar{y}\|_2^2} \leq \sqrt{\bar{\delta}} \leq \frac{1}{2}\|u - l\|_2, \tag{65}$$

where the second inequality follows from the fact that $\sqrt{a} - \sqrt{b} \leq \sqrt{a - b}$ for $a \geq b \geq 0$. It then follows from (63) and (65) that (17) holds true. □

Proof of Corollary 1 By Theorem 3, we have

$$0 \geq f_2(\bar{y}) - f^*_{[l,u]} \geq f_2(\bar{y}) - v^*_{[l,u]} \geq -\sqrt{\sum_{i=1}^r [\bar{s}_i - \bar{z}_i^2]} \geq -\epsilon.$$

Thus, $f^*_{[l,u]} \geq f_2(\bar{y}) \geq f^*_{[l,u]} - \epsilon$ and so $\bar{y}$ is an ϵ-optimal solution of problem (17). □

Proof of Lemma 7 Since (y^k, t^k, s^k, z^k) is the optimal solution of problem (15) over $[l = l^k, u = u^k]$, by Theorem 3, it follows that $t^k - f_2(y^k) \leq \epsilon$. From Steps 2 and (S3.5) of the algorithm, we see that $f_2(y^k) \leq v^* = f_2(y^*)$ for all k. Thus $t^k - v^* \leq t^k - f_2(y^k) \leq \epsilon$. This means that the stopping criterion is satisfied, so the algorithm stops. By Step (S3.1),

t^k is the largest upper bound. Thus $f_2^* \leq t^k$, where f_2^* denotes the optimal value of problem (8). Note that both y^* and y^k are feasible solutions to problem (8). Therefore,

$$f_2(y^k) \leq f_2^* \leq t^k \leq f_2(y^k) + \epsilon, \quad f_2(y^*) \leq f_2^* \leq t^k \leq v^* + \epsilon = f_2(y^*) + \epsilon,$$

which imply that both y^k and y^* are global ϵ-solutions to problem (8). □

Proof of Theorem 5 At the k-th iteration, if the chosen node $[\mathcal{B}^k, (y^k, t^k, s^k, z^k)]$ with the largest upper bound t^k satisfies either $u_{i^*}^k - l_{i^*}^k \leq 2\epsilon/\sqrt{r}$ for the chosen i^* in partition or $\|u^k - l^k\|_\infty \leq 2\epsilon/\sqrt{r}$, then, by Lemmas 6 and 7, the algorithm stops and yields a global ϵ-approximate solution y^* to problem (8). At the k-th iteration, if the algorithm does not stop, then, by Lemma 7, $s_{i^*}^k - (t_{i^*}^k)^2 > \epsilon^2/r$ for the chosen i^* in Step (S3.3) and hence $u_{i^*}^k - l_{i^*}^k > 2\epsilon/\sqrt{r}$ by Lemma 6. That is, the i^*-th edge of sub-rectangle $\mathcal{B}^k$ must be longer than $2\epsilon/\sqrt{r}$. According to Step (S3.3), it will be divided at either point $z_{i^*}^k$ or the midpoint of this edge. Note that if $u_i^k - l_i^k \leq 2\epsilon/\sqrt{r}$, then the i-th direction will never be chosen in Step (S3.3) as a branching direction at the current iteration. This implies that all the edges of sub-rectangle corresponding to a node with the largest upper bound chosen at each iteration will never be shorter than $2\epsilon/\sqrt{r}$. Thus, every edge of the initial rectangle will be divided into at most $\left\lceil \frac{\sqrt{r}(t_u^i - t_l^i)}{2\epsilon} \right\rceil$ sub-intervals. In other words, to obtain an ϵ-optimal solution to problem (8), the total number of the relaxed subproblem (15) required to be solved in all the runs of Algorithm 2 is bounded above by

$$\prod_{i=1}^{r} \left\lceil \frac{\sqrt{r}(z_u^i - z_l^i)}{2\epsilon} \right\rceil.$$

This completes the proof. □

Proof of Proposition 10 Given any $u \in \mathcal{B}_p$, we consider the following two LO problems,

$$z_1(u) \quad = \quad \min_{x \in \mathcal{F}_1(u)} - e^T x, \tag{66}$$

$$z_2(u) \quad = \quad \min_{x \in \mathcal{F}_2(u)} - e^T x. \tag{67}$$

It suffices to prove the equivalence between (66) and (67). The proof of this claim is a minor modification of the proof of Theorem 2.1 in [4]. For self-completeness, we give the detailed proof here. Note that $\mathcal{F}_1(u) \subset \mathcal{F}_2(u)$. Since $\mathcal{F}_1(u)$ is nonempty as assumed, it follows that $\mathcal{F}_2(u)$ is also nonempty. Let $\bar{x}(u)$ be the vector whose element $\bar{x}_i(u)$ is defined by

$$\bar{x}_i(u) := \sup\{x_i : x \in \mathcal{F}_2(u)\}, \quad i = 1, \ldots, n.$$

From the definition of $\mathcal{F}_2(u)$, $\bar{x}(u) \leq e$. Since $\mathcal{F}_1(u) \subset \mathcal{F}_2(u)$, it follows that

$$\bar{x}_i(u) = \sup\{x_i : x \in \mathcal{F}_2(u)\} \geq \sup\{x_i : x \in \mathcal{F}_1(u)\} \geq 0, \quad i = 1, \ldots, n.$$

On the other hand, for each $i = 1, \ldots, n$, there exists $x^i \in \mathcal{F}_2(u)$ such that $\bar{x}_i(u) = x^i_i$. Thus,

$$\left(\sum_{j=1}^{n} L_{ij}\right) \bar{x}_i(u) - \sum_{j=1}^{n} L_{ji} x^i_j = \left(\sum_{j=1}^{n} L_{ij}\right) x^i_i - \sum_{j=1}^{n} L_{ji} x^i_j, \quad i = 1, \ldots, n. \tag{68}$$

Since $L_{ij} \geq 0$ for $i, j = 1, \ldots, n$, we have

$$\left(\sum_{j=1}^{n} L_{ij}\right) \bar{x}_i(u) - \sum_{j=1}^{n} L_{ji} \bar{x}_j(u) \leq \left(\sum_{j=1}^{n} L_{ij}\right) \bar{x}_i(u) - \sum_{j=1}^{n} L_{ji} \bar{x}^i_j, \quad i = 1, \ldots, n. \tag{69}$$

By using $x^i \in \mathcal{F}_2(u)$, $i = 1, \ldots, n$, we can infer from (68) and (69) that

$$\left(\sum_{j=1}^{n} L_{ij}\right) \bar{x}_i(u) - \sum_{j=1}^{n} L_{ji} \bar{x}_j(u) \leq b_i(u), \quad i = 1, \ldots, n,$$

where $b_i(u)$ denotes the i-th component of vector $b(u) = Qu + \hat{b}$. Thus, we have $\bar{x}(u) \in \mathcal{F}_2(u)$ and $\bar{x}(u) \in \mathcal{F}_1(u)$. On the other hand, by the definition of $\bar{x}(u)$, it is easy to see that

$$-e^T x \geq -e^T \bar{x}(u), \quad \forall x \in \mathcal{F}_2(u),$$

so $\bar{x}(u)$ is an optimal solution of problem (67). Thus, using $\mathcal{F}_1(u) \subset \mathcal{F}_2(u)$ and $\bar{x}(u) \in \mathcal{F}_1(u)$, we have

$$-e^T \bar{x}(u) = z_2(u) \leq z_1(u) \leq -e^T \bar{x}(u) \quad \Longrightarrow \quad z_1(u) = z_2(u) = -e^T \bar{x}(u),$$

so $\bar{x}(u)$ is also an optimal solution of problem (66). □

Proof of Proposition 11 Given $u \in \mathcal{B}_p$, we consider the following LO problems,

$$g_1(u) = \min_{z \in \mathcal{F}(u)} e^T z, \tag{70}$$

$$g_2(u) = \min_{(z,v) \in \mathcal{F}_0(u)} e^T z + Mv. \tag{71}$$

For all $z \in \mathcal{F}(u)$, it is oblivious that $(z, 0) \in \mathcal{F}_0(u)$. Let $(z^*(u), v^*(u))$ be the optimal solution of problem (71). Then,

$$e^T z^*(u) + Mv^*(u) \leq e^T z \quad \Rightarrow \quad Mv^*(u) \leq e^T (z - z^*(u)),$$

which implies $v^*(u) = 0$ since $M > 0$ is sufficiently large. Thus, $(z^*(u), v^*(u)) \in \mathcal{F}_0(u)$ implies $z^*(u) \in \mathcal{F}(u)$. Moreover, we have $e^T z^*(u) \leq e^T z$ for any $z \in \mathcal{F}(u)$. This proves that $z^*(u)$ is the optimal solution of problem (70) with $g_1(u) = e^T z^*(u)$. Note that, since $v^*(u) = 0$, we have $g_2(u) = e^T z^*(u)$. Thus $g_1(u) = g_2(u) = e^T z^*(u)$ and then problems (31) and (30) have the same optimal value. □

Proof of Proposition 13 From (33), $\psi_q(z)$ is the maximum of linear functions of z. Thus, $\psi_q(z)$ is a piecewise linear convex function on $\mathbb{R}^l$. Since y_z is a globally optimal solution of problem (33), we obtain $\psi_q(z) = \|Q^T y_z\|_q + (b_0 - Dz)^T y_z$. For any $\mu \in \mathbb{R}^l$, we have

$$\begin{aligned}\psi_q(\mu) &= \max_{y\in\mathcal{C}} \left\{\|Q^T y\|_q + (b_0 - D\mu)^T y\right\} \\ &\geq \|Q^T y_z\|_q + (b_0 - D\mu)^T y_z \\ &= \|Q^T y_z\|_q + (b_0 - Dz)^T y_z + (-D^T y_z)^T(\mu - z) \\ &= \psi_q(z) + (-D^T y_z)^T(\mu - z).\end{aligned}$$

Thus, $-D^T y_z$ is a subgradient of $\psi_q(z)$ at z. □

Proof of Lemma 8 We see from Step 0 and (33) that $\mathcal{R} \subseteq \mathcal{P}_1$. Suppose that $\mathcal{R} \subseteq \mathcal{P}_l$, $l \geq 1$. By Step 1, (z^l, η_l) is an optimal solution to problem (35) with $k = l$. By Step 2, y^{l+1} is the optimal solution of problem (33) with $z = z^l$, and $\psi_q(z^l) = \|Q^T y^{l+1}\|_q + (b_0 - Dz^l)^T y^{l+1}$. By Proposition 13, $\psi_q(z)$ is convex and $-D^T y^{l+1}$ is a subgradient of $\psi_q(z)$ at z^l. Thus, for any $(z, \eta) \in \mathcal{R} \subseteq \mathcal{P}_l$, we have

$$\eta \geq \psi_q(z) \geq \psi_q(z^l) + (-D^T y^{l+1})^T(z - z^l) = \|Q^T y^{l+1}\|_q + (b_0 - Dz)^T y^{l+1},$$

which, together with $(z, \eta) \in \mathcal{P}_l$, implies that $(z, \eta) \in \mathcal{P}_{l+1}$ and so $\mathcal{R} \subseteq \mathcal{P}_{l+1}$. We conclude, by induction, that $\mathcal{R} \subseteq \mathcal{P}_k$ for all k. □

Proof of Theorem 7 Let $\{(z^k, \eta_k)\}$ be an infinite sequence generated by Algorithm 5, and let $(\bar{z}, \bar{\eta})$ be its accumulation point. Then there exists a subsequence $\mathcal{K} \subset \{1, 2, \ldots\}$ such that $\{(z^k, \eta_k)\}_{\mathcal{K}} \to (\bar{z}, \bar{\eta})$. The closedness of $\mathcal{Z}$ and $\{z^k\} \subseteq \mathcal{Z}$ imply $\bar{z} \in \mathcal{Z}$. By Step 1, (z^k, η_k) is an optimal solution to problem (35) for any k. Then, for any $k', k \in \mathcal{K}$, when $k' \geq k + 1$, we must have

$$\begin{aligned}\eta_{k'} &\geq \|Q^T y^{k+1}\|_q + (b_0 - Dz^{k'})^T y^{k+1} \\ &= \|Q^T y^{k+1}\|_q + (b_0 - Dz^k)^T y^{k+1} - (D^T y^{k+1})^T(z^k - z^{k'}) \\ &= \psi_q(z^k) - (D^T y^{k+1})^T(z^k - z^{k'}) \\ &\geq \psi_q(z^k) - \|D^T y^{k+1}\|\|z^k - z^{k'}\|,\end{aligned}$$

where the second equality follows from the fact that y^{k+1} is the optimal solution to problem (33) with $z = z^k$. By Assumption 1, the sequence $\{y^k\}$ is bounded and hence $\{\|D^T y^{k+1}\|\}$ is bounded. Taking the limit in the above inequality as $k', k \in \mathcal{K} \to \infty$ gives rise to $\bar{\eta} \geq \psi_q(\bar{z})$.

Let f^* be the optimal value of problem (32). Note that problem (32) is equivalent to problem $\min\{d^T z + \eta : (z, \eta) \in \mathcal{R}\}$ in the sense that they have the same optimal value. By Lemma 8, $\mathcal{R} \subseteq \mathcal{P}_k$ for all k. Since (z^k, η_k) is an optimal solution to problem (35), we have

$f^* \geq d^T z^k + \eta_k$ for all k. Taking the limit as $k \in \mathcal{K} \to \infty$ yields $f^* \geq d^T \bar{z} + \bar{\eta}$. Note that since $\bar{z} \in \mathcal{Z}$ and $\bar{\eta} \geq \psi_q(\bar{z})$, we follow that

$$d^T \bar{z} + \psi_q(\bar{z}) \geq f^* \geq d^T \bar{z} + \bar{\eta} \geq d^T \bar{z} + \psi_q(\bar{z}).$$

Thus, $f^* = d^T \bar{z} + \psi_q(\bar{z})$ and so $\bar{z}$ is an optimal solution of problem (32). □

Proof of Lemma 9 We consider the k-th iteration. By Step 1, $(z^k, \eta_k, \bar{x}^1, \ldots, \bar{x}^k)$ is an optimal solution of problem (36). Then, $f_k = d^T z^k + \eta_k$, and

$$\begin{cases} \eta_k \geq c^T \bar{x}^i, \quad i = 1, \ldots, k, \\ A\bar{x}^i + Dz^k \leq Qu_q^i + b_0, \quad i = 1, \ldots, k, \\ z^k \in \mathcal{Z}, \quad \bar{x}^i \geq 0, \quad i = 1, \ldots, k. \end{cases} \tag{72}$$

By Steps 0-(ii) and 2, y^i is an optimal solution of problem problem (33) with $z = z^{i-1}$ for $i = 1, \ldots, k$. For each $i = 1, \ldots, k$, if $q \in (1, \infty)$, by Step 1, $u_q^i = \mu(y^i; q)$, where $\mu(y^i, q)$ is given in (1). Since $\frac{1}{p} + \frac{1}{q} = 1$, via simple calculus we obtain from (1) that

$$\|u_q^i\|_p = 1, \qquad (u_q^i)^T Q^T y^i = \|Q^T y^i\|_q, \quad i = 1, \ldots, k. \tag{73}$$

If $q = 1$, by Step 1, u_q^i is derived by (4) with $y^* = y^i$. For this case, it is easy to check that (73) holds. Note that since $y^i \in \mathcal{C}$ for $i = 1, \ldots, k$, we have $A^T y^i \leq c$ and $y^i \leq 0$ for $i = 1, \ldots, k$. It then follows from (72) and (73) that

$$\begin{aligned} \eta_k \geq c^T \bar{x}^i &\geq (y^i)^T A\bar{x}^i \geq (y^i)^T Qu_q^i + (b_0 - Dz^k)^T y^i \\ &= \|Q^T y^i\|_q + (b_0 - Dz^k)^T y^i, \quad i = 1, \ldots, k, \end{aligned}$$

which, by $z^k \in \mathcal{Z}$, implies that $(\eta_k, z^k) \in \mathcal{P}_k$, where $\mathcal{P}_k$ denotes the set of feasible solutions of problem (35).

Let z_p^* be an optimal solution to (TSARO$_p$) and f_p^* be its optimal value. For any $\tilde{b} \in \mathcal{U}_p$, let $x^*(\tilde{b})$ be an optimal solution to the LO problem

$$\phi(z_p^*, \tilde{b}) := \min_{x \in \mathcal{X}(z_p^*, \tilde{b})} c^T x.$$

Then, $x^*(\tilde{b})$ satisfies $Ax^*(\tilde{b}) + Dz_p^* \leq \tilde{b}$ and $x^*(\tilde{b}) \geq 0$. Define $\eta^* := \max_{\tilde{b} \in \mathcal{U}_p} \phi(z_p^*, \tilde{b})$. We have $f_p^* = d^T z_p^* + \eta^*$ and $\eta^* \geq \phi(z_p^*, \tilde{b}) = c^T x^*(\tilde{b})$ for any $\tilde{b} \in \mathcal{U}_p$.

Let $b^i = Qu_q^i + b_0$, $i = 1, \ldots, k$. From (73), we see that $b^i \in \mathcal{U}_p$ for $i = 1, \ldots, k$. It then follows that $(z_p^*, \eta^*, x^*(b^1), \ldots, x^*(b^k))$ is a feasible solution to problem (36) and the corresponding objective value is $f_p^* = d^T z_p^* + \eta^*$, which yields $f_p^* \geq f_k = d^T z^k + \eta_k$. □

Proof of Theorem 8 Let $\{(z^k, \eta_k)\}$ be an infinite sequence generated by the algorithm, and let $(\bar{z}, \bar{\eta})$ be its accumulation point. Then there exists a subsequence $\mathcal{K} \subset \{1, 2, \ldots\}$ such that $\{(z^k, \eta_k)\}_{\mathcal{K}} \to (\bar{z}, \bar{\eta})$. The closedness of $\mathcal{Z}$ and $\{z^k\} \subseteq \mathcal{Z}$ imply $\bar{z} \in \mathcal{Z}$. By Lemma 9, $(z^k, \eta^k) \in \mathcal{P}_k$ for all k, i.e,

$$\eta_k \geq \|Q^T y^i\|_q + (b_0 - Dz^k)^T y^i,\ i = 1, \ldots, k$$

for all k. Using the similar arguments as in the proof of Theorem 7, we can follow from the above inequality that $\bar{\eta} \geq \psi_q(\bar{z})$.

Let f^* be the optimal value of problem (32). Note that problem (32) and (TSARO$_p$) have the same optimal value. By Lemma 9, $f^* \geq d^T z^k + \eta_k$ for all k. Using the similar arguments as in the proof of Theorem 7, we can prove that $f^* = d^T \bar{z} + \psi_q(\bar{z})$ and so $\bar{z}$ is an optimal solution of problem (32). □

We explain in the following counter example that the NLSDP problem (39) may not necessarily find the globally optimal solution of the WCSR problem (27) with $p = 2$.

Example 1.1. Consider the following instance of the WCSR problem (27) with $p = 2$ of the size $n = 5$ and $r = 5$:

$$L = \begin{pmatrix} 0 & 0.6574 & 6.9739 & 0.3202 & 0.6223 \\ 0.3275 & 0 & 0.6970 & 1.5259 & 1.1412 \\ 1.7818 & 0.1406 & 0 & 3.7880 & 3.4084 \\ 4.9287 & 2.3899 & 0.3288 & 0 & 1.1622 \\ 0.1895 & 0.6807 & 0.5516 & 1.5942 & 0 \end{pmatrix}, \quad \hat{b} = \begin{pmatrix} 0.1078 \\ 1.0250 \\ 4.0447 \\ 2.0990 \\ 6.3281 \end{pmatrix},$$

$$Q = \begin{pmatrix} -1.2046 & 1.4553 & -1.2589 & -0.2527 & 1.4621 \\ -0.2844 & -0.4540 & -1.0949 & -1.5571 & -2.3645 \\ -0.4641 & -0.4239 & 0.9221 & 1.5834 & 1.0894 \\ -0.8424 & -2.2508 & 2.7416 & 1.5560 & -0.2204 \\ 0.3499 & -2.8534 & 2.6144 & 1.4439 & -1.7270 \end{pmatrix}.$$

Note that by Proposition 10, two problems (27) and (31) have the same optimal value. Using the BSA algorithm in [3] to solve the NLSDP problem (39) with $\bar{M} = 10$, we can obtain the lower bound $LB = 0.5329$ and upper bound $UB = 0.5600$. However, solving problem (31) with $A = L^T - \mathrm{diag}(Le)$ and $b_0 = Ae + \hat{b}$ by using the SCOBB algorithm, the optimal value of this instance is 0.5583. This example then illustrates that we may not be able to find the global optimal solution to the WCSR problem (27) with $p = 2$ via only solving its NLSDP relaxation (39).

2 Mixed Integer Program Reformulation for WCSR

In this section, we describe the mixed integer program reformulation proposed in [5] for the WCSR problem (27) with $p = 2, \infty$. Note that problems (27) and (29) are equivalent due to

Proposition 10. By using KKT conditions, problem (29) with $p = 2, \infty$ can be reformulated as an equivalent LO problem with complementary constraints:

$$\begin{aligned} \max_{z,y,u} \quad & e^T z \\ \text{s.t.} \quad & Az \leq Qu + b_0, \quad A^T y \leq e, \\ & y^T(Qu + b_0 - Az) = 0, \quad z^T(c - A^T y) = 0, \\ & z \geq 0, \quad y \leq 0, \quad \|u\|_p \leq 1, \end{aligned} \tag{74}$$

where $p = 2, \infty$, $A = L^T - \text{diag}(Le)$, $b_0 = Ae + \hat{b}$. By making use of the big-M method, complementary constraints in (74) can be linearized by introducing binary variables. For example, by introducing binary variables $w \in \{0, 1\}^n$, we can reformulate complementary constraint $y^T(Qu + b_0 - Az) = 0$ as the following:

$$y \geq -Mw, \quad Qu + b_0 - Az \leq M(e - w). \tag{75}$$

Then problem (29) can be converted into the following 0-1 mixed integer program:

$$\begin{aligned} \max_{z,y,u,v,w} \quad & e^T z \\ \text{s.t.} \quad & Az \leq Qu + b_0, \quad A^T y \leq e, \\ & y \geq -Mw, \quad Qu + b_0 - Az \leq M(e - w), \\ & z \leq Mv, \quad e - A^T y \leq M(e - v), \\ & z \geq 0, \quad y \leq 0, \quad \|u\|_p \leq 1, \\ & w \in \{0, 1\}^n, \quad v \in \{0, 1\}^n, \end{aligned} \tag{76}$$

which can be solved by the existing solver. Here, $p = 2, \infty$, $A = L^T - \text{diag}(Le)$, $b_0 = Ae + \hat{b}$, and $M > 0$ is sufficiently large penalty parameter.

3 Numerical results of the SCO-NLSDR algorithm

In this subsetion, we test the SCO-NLSDR algorithm (Algorithm 3) for (WCLO_2) when $r > 10$. The data (A, Q, b_0, c) in (WCLO_2) are randomly generated in the same fashion as in [3]. That is, each entry of A and b_0 is drawn from $U(-5, 5)$ (i.e., uniformly distributed within interval $[-5, 5]$), each entry of Q is drawn from $U(-2, 2)$ and each entry of c is drawn from $U(0, 1)$. The non-negativity constraint on x in (WCLO_2) and the choice of non-negative vector c make the randomly generated problem feasible and bounded in most cases.

To measure the effectiveness of the algorithm, we define the reduced ratio of the gap as

$$\text{reduc.ratio} = \left(1 - \frac{UB_1 - LB_1}{UB_0 - LB_0}\right) \times 100\%, \tag{77}$$

where $UB_0 - LB_0$ and $UB_1 - LB_1$ denote the gap between the lower and upper bounds derived from NLSDP and Algorithm 3, respectively. Here, NLSDP denotes the nonlinear SDP relaxation (18). In Tables 1 and 2, we summarize the numerical results of Algorithm 3, SCOBB and NLSDP for 10 random instances of ($WCLO_2$) of the same size for the cases $r = 5, 15, 20, 30, 40$. From Tables 1 and 2, we see that the solutions from SCO are optimal for most test problems. The smaller gap between the lower and upper bounds in Algorithm 3 also indicates its outperformance.

Table 1: Comparison results of Algorithm 3 VS NLSDP for ($WCLO_2$) with $n = 20, m = 8, r = 5$

	SCOBB	NLSDP			Algorithm 3				
Prob.	Opt.val	LB_0	UB_0	Time	LB_1	UB_1	Time	Iter	reduc.ratio
1	3.095533	3.059959	3.096726	3.6798	$*$3.095533	3.096374	13.4220	2	97.7%
2	0.532110	0.527029	0.533068	4.1089	$*$0.532110	0.532643	14.4159	2	91.2%
3	1.035612	1.032725	1.052760	3.0691	$*$1.035612	1.052760	3.6807	0	14.4%
4	0.858580	0.833858	0.866187	3.8223	$\star$ 0.858505	0.866187	5.1331	0	76.2%
5	0.257205	0.246188	0.257872	2.9501	$*$0.257205	0.257872	4.8460	0	94.3%
6	2.964924	2.962652	2.965273	2.9916	$*$2.964924	2.965273	3.6527	0	86.7%
7	0.709320	0.707287	0.712555	2.6539	$*$0.709320	0.711952	6.6721	1	50.0%
8	1.719650	1.718029	1.720099	3.0937	$*$1.719650	1.719908	8.8180	1	87.6%
9	0.392790	0.389372	0.395454	4.0207	$*$0.392790	0.395454	5.3288	0	56.2%
10	1.082983	1.045696	1.086230	3.6187	$*$1.082983	1.086210	24.6361	4	92.0%

Note: The sign "$*$" denotes the instance in which the global optimal solution is found by SCO. The sign "$\star$" denotes the one in which SCO obtains an ϵ-optimal solution with $\epsilon \leq 10^{-4}$.

References

[1] Garey, M. R., D. S. Johnson. 1978. Strong NP-Completeness: results, motivation examples and implications. *J. Assoc. Comput. Mach.*, **25(3)** 499–508.

[2] Ge, D., X. Jiang, Y. Ye. 2011. A note on the complexity of ℓ_p minimization. *Math. Program.* , **129(2)** 285–299.

[3] Peng, J., Z. Tao. 2015. A nonlinear semidefinite optimization relaxation for the worst-case linear optimization under uncertainties. *Math. Program.* **152(1)** 593–614.

[4] Pokutta, S., C. Schmaltz, S. Stiller. 2011. Measuring systemic risk and contagion in financial networks. *Available at SSRN 1773089.*

Table 2: Average performance of Algorithm 3 for 10 random instances of ($WCLO_2$) with $r = 15, 20, 30, 40$

Size			SCOBB		NLSDP			Algorithm 3				
n	m	r	Opt.val	Time	LB_0	UB_0	Time	LB_1	UB_1	Time	Iter	reduc.ratio
30	20	15	4.140961	128.24	4.024888	4.162921	8.60	4.128135[7]	4.162824	17.53	0.1	74.9%
30	20	20	4.604469	667.20	4.580749	4.610568	7.64	4.604289[8]	4.610567	12.95	0.1	78.9%
40	30	15	4.135156	742.33	4.036657	4.167734	20.66	4.132854[6]	4.167063	100.72	1.6	73.9%
40	30	20	5.676958(5)	2150.19	5.604147	5.690827	30.57	5.676413[5]	5.690806	71.13	0.1	83.4%
50	30	20	2.968602(6)	2635.52	2.863048	2.997755	25.62	2.962316[3]	2.997694	75.35	0.3	73.7%
50	30	30	4.486044(8)	3334.04	4.398321	4.507435	21.58	4.488480[2]	4.507397	71.22	0.3	82.7%
60	40	20	6.178780(6)	2761.65	6.127152	6.195050	53.67	6.178780[4]	6.195013	159.14	0.2	76.1%
60	40	30	9.332768(8)	3052.99	9.114029	9.380993	52.93	9.324228[2]	9.380993	137.75	0.0	78.7%
60	40	40	7.201462(10)	3600	6.966191	7.262731	68.80	7.191342[0]	7.262731	170.48	0.0	75.9%

Note: The number in parentheses in the column of SCOBB stands for the number of instances in which SCOBB only reports the best feasible solution obtained within 3600 seconds and fails to verify the global optimality of the obtained solution. The number in bracket in the column of Algorithm 3 stands for the number of instances in which the solution derived by SCO is the global optimal solution.

[5] Zeng, B., L. Zhao. 2013. Solving two-stage robust optimization problems using a column-and-constraint generation method. *Oper. Res. Lett.* **41** 457–461.