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# SIMULTANEOUS DIAGONALIZATION OF MATRICES AND ITS APPLICATIONS IN QUADRATICALLY CONSTRAINED QUADRATIC PROGRAMMING*

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**Abstract.** An equivalence between attainability of simultaneous diagonalization (SD) and hidden convexity in quadratically constrained quadratic programming (QCQP) stimulates us to investigate necessary and sufficient SD conditions, which is one of the open problems posted by Hiriart-Urruty [*SIAM Rev.*, 49 (2007), pp. 255–273] nine years ago. In this paper we give a necessary and sufficient SD condition for any two real symmetric matrices and offer a necessary and sufficient SD condition for any finite collection of real symmetric matrices under the existence assumption of a semidefinite matrix pencil. Moreover, we apply our SD conditions to QCQP, especially with one or two quadratic constraints, to verify the exactness of its second-order cone programming relaxation and to facilitate the solution process of QCQP.

**Key words.** simultaneous diagonalization, congruence, quadratically constrained quadratic programming, second-order cone programming relaxation

**AMS subject classifications.** 15A, 65K, 90C

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## 1. Introduction.

Ben-Tal and Hertog [2] show that if the two matrices in the objective function and the constraint of a quadratically constrained quadratic programming (QCQP) problem are simultaneously diagonalizable (SD), this QCQP problem can then be transformed into an equivalent second-order cone programming (SOCP) problem, which can be solved much faster than the semidefinite programming (SDP) reformulation. (Note that to simplify the notation, we use the abbreviation SD to denote both "simultaneously diagonalizable" and "simultaneous diagonalization" via congruence when no confusion will arise. We also assume that all the matrices in this paper are real.) A natural question to ask is when a given pair of matrices is SD. Essentially, the SD problem of a finite collection of symmetric matrices via congruence is one of the 14 open problems posted by Hiriart-Urruty [6] nine years ago: "A collection of $m$ symmetric $(n, n)$ matrices $\{A_1, A_2, \ldots, A_m\}$ is said to be *simultaneously diagonalizable via congruence* if there exists a nonsingular matrix $P$ such that each of the $P^T A_i P$ is diagonal."

Two sufficient conditions for SD of two matrices $A$ and $B$ have been developed in [6, 13, 14, 17]: (i) There exist $\mu_1$ and $\mu_2 \in \Re$ such that $\mu_1 A + \mu_2 B \succ 0$, which is termed as the *regular case* in [13, 14, 17]; and (ii) when $n \geq 3$, ($\langle Ax, x\rangle = 0$ and $\langle Bx, x\rangle = 0$) implies ($x = 0$). Lancaster and Rodman [10] offer another sufficient condition for SD of two matrices $A$ and $B$: There exist $\alpha$ and $\beta$ such that $C := \alpha A + \beta B \succeq 0$ and

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$Ker(C) \subseteq Ker(A) \cap Ker(B)$. Several other SD sufficient conditions of two matrices are presented in Theorem 4.5.15 in [7]. A necessary and sufficient condition for SD of two matrices has been derived by Uhlig in [20, 21] if at least one of the two matrices is nonsingular. A sufficient SD condition is well known for multiple matrices: If $m$ matrices commute with each other, then they are SD; see [7, Problems 22 and 23, page 243]. The equivalence between the attainability of SD and a hidden convexity in QCQP stimulates the research in this paper.

In this paper, we provide a necessary and sufficient SD condition for any two symmetric matrices, which extends the results in [21] for any two arbitrary matrices. We then show its applications in QCQP with one quadratic constraint or with an interval constraint. Furthermore, we find a necessary and sufficient SD condition for $m$ ($m \geq 3$) symmetric matrices when there is a semidefinite matrix pencil, i.e., there exist $\lambda_1, \ldots, \lambda_m \in \Re$ (not all of which are zero), such that $\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_m A_m \succeq 0$. The semidefinite matrix pencil assumption has been a widely adopted assumption in the solvability study in SDP and its duality [9, 18]. We also show its applications in QCQP and show that QCQP with two quadratic constraints has a nontrivial exact relaxation. Our proofs are constructive for both cases of two matrices and multiple (more than two) matrices. Thus, we essentially establish systematic procedures for testing SD using the derived sufficient and necessary condition for any finite number of symmetric matrices.

Our paper is organized as follows. In section 2, we provide a necessary and sufficient SD condition for any pair of symmetric matrices and show its applications in QCQP with one quadratic constraint or an interval quadratic constraint. In section 3, we show necessary and sufficient SD conditions for a finite number of symmetric matrices under the definite matrix pencil condition and a more general semidefinite matrix pencil condition. We next discuss the applications of SD in QCQP with multiple, in particularly two, quadratic constraints in section 4. Finally, we conclude the paper in section 5.

**Notation.** We use $I_p$ and $0_p$ to denote the identity matrix of dimension $p \times p$ and the zero square matrix of dimension $p \times p$, respectively, and use $0_{p\times q}$ to denote a zero matrix of dimension $p \times q$. The notation $\Re^n$ represents the $n$ dimensional real vector space, and $S^n$ represents the $n \times n$ real symmetric matrix space. We denote by $tr(A)$ the trace of a square matrix $A$, and by $\operatorname{diag}(A_1, A_2, \ldots, A_k)$ the block diagonal matrix

$$\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix},$$

where $A_i$, $i = 1, \ldots, k$, are all square matrices. Finally, we use $\operatorname{Diag}(X)$ to denote the vector formed by the diagonal elements of the square matrix $X$ with $\operatorname{Diag}(X)_k$ representing its $k$th diagonal entry.

## 2. Simultaneous diagonalization for two matrices.

In this section, we first derive a necessary and sufficient SD condition for any two matrices and then show its applications in the trust region subproblem (TRS) and its variants.

### 2.1. SD condition.

The following lemma from [21] shows a necessary and sufficient SD condition of two matrices if at least one matrix is nonsingular.

LEMMA 1 (see [21]). *If one of the two symmetric matrices $A$ and $B$ is nonsingular (without loss of generality, we assume that $A$ is nonsingular), they are SD if and* *only if the Jordan normal form of $A^{-1}B$ is a diagonal matrix.*

The following lemma is also useful in deriving the congruent matrix to achieve the SD for two matrices.

LEMMA 2 (see [20, Lemma 1]). *Let a Jordan matrix be denoted by*

$$K := \operatorname{diag}(C(\lambda_1), C(\lambda_2), \dots, C(\lambda_k)),$$

*where* $C(\lambda_i) := \operatorname{diag}(K_{i_1}(\lambda_i), K_{i_2}(\lambda_i), \dots, K_{i_t}(\lambda_i))$ *denotes all the Jordan blocks associated with eigenvalue* $\lambda_i$*, and*

$$K_{i_j} := \begin{pmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{pmatrix}_{i_j \times i_j},$$

$j = 1, 2, \dots, t$.

*For a symmetric matrix $S$, if $SK$ is symmetric, then $S$ is block diagonal, and* $S = \operatorname{diag}(S_1, S_2, \dots, S_k)$ *with* $\dim S_i = \dim C(\lambda_i)$.

When the condition in Lemma 1 is satisfied, we can identify a congruent matrix that makes two $n \times n$ symmetric matrices $A$ and $B$ SD as follows.

When $A$ is nonsingular and the Jordan normal form, denoted by $J$, of $A^{-1}B$ is diagonal, there exists a nonsingular matrix $P$ such that

$$J := P^{-1}A^{-1}BP = \operatorname{diag}(\lambda_1 I_{n_1}, \dots, \lambda_k I_{n_k}),$$

where $k \leq n$. Note that $P^TAP$ is symmetric and $(P^TAP)J = P^TBP$ is symmetric. Applying Lemma 2, we have the following two block diagonal matrices:

$$P^TAP = \operatorname{diag}(A_1, \dots, A_k) \text{ and } P^TBP = \operatorname{diag}(B_1, \dots, B_k),$$

where $\operatorname{diag}(A_1, \dots, A_k)$ and $\operatorname{diag}(B_1, \dots, B_k)$ have the same partition as $J$, i.e., $\dim A_i = \dim B_i = \dim I_{n_i}$, $i = 1, \dots, k$. Moreover,

$$\begin{aligned} J &= P^{-1}A^{-1}BP \\ &= (P^TAP)^{-1}P^TBP \\ &= (\operatorname{diag}(A_1, \dots, A_k))^{-1} \operatorname{diag}(B_1, \dots, B_k) \\ &= \operatorname{diag}(A_1^{-1}B_1, \dots, A_k^{-1}B_k). \end{aligned}$$

Thus, for $i = 1, \dots, k$, $A_i^{-1}B_i = \lambda_i I_{n_i}$, i.e., $\lambda_i A_i = B_i$. Define $R := \operatorname{diag}(R_1, \dots, R_k)$, where $R_i^T A_i R_i$ is a spectral decomposition of $A_i$, $i = 1, \dots, k$. Thus $R_i^T B_i R_i = \lambda_i R_i^T A_i R_i$ is also diagonal, $i = 1, \dots, k$. So with the two nonsingular real matrices $P$ and $R$, $(PR)^T APR$ and $(PR)^T BPR$ are both diagonal; i.e., $A$ and $B$ are SD via the congruent matrix $PR$.

LEMMA 3. *Let* $A := \operatorname{diag}(A_1, 0_{q+r})$ and

$$B := \begin{pmatrix} B_1 & 0_{p\times q} & B_2 \\ 0_{q\times p} & B_3 & 0_{q\times r} \\ B_2^T & 0_{r\times q} & 0_r \end{pmatrix},$$

*where both $A_1$ and $B_1$ are $p \times p$ nonsingular symmetric matrices, $B_3$ is a $q \times q$ nonsingular symmetric matrix, and $B_2$ is a $p \times r$ full column rank matrix with $r \leq p$. Then $A$ and $B$ cannot be SD.*

*Proof.* We will prove the lemma by contradiction. Let us assume that $A$ and $B$ are SD. Then, there exists a congruent matrix

$$P := \begin{pmatrix} P_1 & P_2 & P_3 \\ P_4 & P_5 & P_6 \\ P_7 & P_8 & P_9 \end{pmatrix},$$

where the partition of $P$ is the same as the partition of $B$, such that $P^TAP$ and $P^TBP$ are diagonal, and without loss of generality, the nonzero elements of $P^TAP$ are all placed in the first $p$ entries of the diagonal,

$$P^TAP = \begin{pmatrix} P_1^TA_1P_1 & P_1^TA_1P_2 & P_1^TA_1P_3 \\ P_2^TA_1P_1 & P_2^TA_1P_2 & P_2^TA_1P_3 \\ P_3^TA_1P_1 & P_3^TA_1P_2 & P_3^TA_1P_3 \end{pmatrix}.$$

Since $A_1$ and $P_1^TA_1P_1$ are both nonsingular and $P_1$ is nonsingular, $P_1^TA_1$ is nonsingular. As $P^TAP$ is diagonal, then $P_1^TA_1P_2$ and $P_1^TA_1P_3$ are both zero matrices. Thus, $P_2 = 0$ and $P_3 = 0$.

Furthermore, the following transformation holds for matrix $B$:

$$P^TBP = \begin{pmatrix} \begin{matrix} P_1^TB_1P_1 + P_1^TB_2P_7 \\ +P_4^TB_3P_4 + P_7^TB_2^TP_1 \end{matrix} & P_1^TB_2P_8 + P_4^TB_3P_5 & P_1^TB_2P_9 + P_4^TB_3P_6 \\ P_8^TB_2^TP_1 + P_5^TB_3^TP_4 & P_5^TB_3P_5 & P_5^TB_3P_6 \\ P_9^TB_2^TP_1 + P_6^TB_3^TP_4 & P_6^TB_3^TP_5 & P_6^TB_3P_6 \end{pmatrix}.$$

From our assumptions, $B_1$ and $B_3$ are both nonsingular, and $B_2$ is of full column rank, which implies that $B$ is nonsingular. So the diagonal matrix $P^TBP$ must be nonsingular. As $P_5^TB_3P_5$ is nonsingular, $P_5$ is thus nonsingular. Note that $P_5^TB_3P_6 = 0$ implies that $P_6 = 0_{q\times r}$, which further leads to $P_6^TB_3P_6 = 0_r$. This contradicts the nonsingularity of $P^TBP$. ◻

LEMMA 4. *For any two $p\times p$ symmetric matrices $A_1$ and $B_1$, and a $q\times q$ matrix $B_2$, if $A_1$ and $B_2$ are nonsingular and diagonal, and $B_1$ is nonsingular, then $A_1$ and $B_1$ are SD if and only if $A := \mathrm{diag}(A_1, 0_q)$ and $B := \mathrm{diag}(B_1, B_2)$ are SD.*

*Proof.* The "⇒" part: If $A_1$ and $B_1$ are SD via congruent matrix $S_{p\times p}$, then the congruent matrix $\mathrm{diag}(S_{p\times p}, I_q)$ makes $A$ and $B$ SD.

The "⇐" part: If $A$ and $B$ are SD, then there exists a nonsingular matrix

$$P := \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix},$$

such that (i) matrix

$$P^TAP = \begin{pmatrix} P_1^T & P_3^T \\ P_2^T & P_4^T \end{pmatrix} \begin{pmatrix} A_1 & \\ & 0_q \end{pmatrix} \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} = \begin{pmatrix} P_1^TA_1P_1 & P_1^TA_1P_2 \\ P_2^TA_1P_1 & P_2^TA_1P_2 \end{pmatrix}$$

is diagonal, where, without loss of generality, the diagonal matrix $P_1^TA_1P_1$ is assumed to be nonsingular, and (ii) matrix $P^TBP$ is diagonal. Then, $P_1$ must be nonsingular and $P_2$ must be a zero matrix. We can now simplify $P^TBP$ to

$$\begin{pmatrix} P_1^T & P_3^T \\ 0_{q\times p} & P_4^T \end{pmatrix} \begin{pmatrix} B_1 & \\ & B_2 \end{pmatrix} \begin{pmatrix} P_1 & 0_{p\times q} \\ P_3 & P_4 \end{pmatrix} = \begin{pmatrix} P_1^TB_1P_1 + P_3^TB_2P_3 & P_3^TB_2P_4 \\ P_4^TB_2P_3 & P_4^TB_2P_4 \end{pmatrix}.$$

Since the congruent matrix $P$ is nonsingular by definition, $P_4$ is nonsingular. Thus, $P_3 = 0$, due to $P_3^TB_2P_4 = 0$, and $B_2$ is nonsingular. Finally, we conclude that $P_1^TB_1P_1$ is diagonal. So $P_1$ is the congruent matrix that makes $A_1$ and $B_1$ SD. ◻

LEMMA 5. *For any two $n \times n$ singular symmetric matrices $A$ and $B$, there always exists a nonsingular matrix $U$ such that*

$$\tilde{A} := U^T A U = \begin{pmatrix} A_1 & 0_{p\times(n-p)} \\ 0_{(n-p)\times p} & 0_{n-p} \end{pmatrix}, \tag{1}$$

*and*

$$\tilde{B} := U^T B U = \begin{pmatrix} \mathcal{B}_1 & 0_{p\times q} & \mathcal{B}_2 \\ 0_{q\times p} & \mathcal{B}_3 & 0_{q\times r} \\ \mathcal{B}_2^T & 0_{r\times q} & 0_r \end{pmatrix}, \tag{2}$$

*where $p$, $q$, $r \geq 0$, $p+q+r = n$, $A_1$ is a nonsingular diagonal matrix, $A_1$ and $\mathcal{B}_1$ have the same dimension of $p \times p$, $\mathcal{B}_2$ is a $p \times r$ matrix, and $\mathcal{B}_3$ is a $q \times q$ nonsingular diagonal matrix.*

*Proof.* We outline a proof for this lemma. Applying the spectral decomposition to $A$ identifies a congruent matrix $Q_1$ such that $\bar{A} := Q_1^T A Q_1 = \operatorname{diag}(A_1, 0_{n-p})$, where $A_1$ is a $p \times p$ nonsingular diagonal matrix. We express the corresponding $Q_1^T B Q_1$ as

$$\bar{B} := Q_1^T B Q_1 = \begin{pmatrix} B_1 & B_2 \\ B_2^T & B_3 \end{pmatrix},$$

where $B_1$ is of dimension $p \times p$.

If $B_3$ is not diagonal, we can further apply the spectral decomposition to $B_3$ by identifying a congruent matrix $Q_2 := \operatorname{diag}(I_p, V_1)$ such that $V_1^T B_3 V_1$ is diagonal and the nonzero eigenvalues of $V_1^T B_3 V_1$ are arranged to the upper left part of its diagonal. Then we have $\hat{A} := Q_2^T \bar{A} Q_2 = \bar{A}$ and

$$\hat{B} := Q_2^T \bar{B} Q_2 = \begin{pmatrix} B_1 & B_4 & B_5 \\ B_4^T & B_6 & 0_{q\times(n-p-q)} \\ B_5^T & 0_{(n-p-q)\times q} & 0_{n-p-q} \end{pmatrix},$$

where $B_6$ is a nonsingular diagonal matrix of dimension $q \times q$.

Applying the congruent matrix

$$Q_3 := \begin{pmatrix} I_p & 0_{p\times q} & 0_{p\times(n-p-q)} \\ -B_6^{-1}B_4^T & I_q & 0_{q\times(n-p-q)} \\ 0_{(n-p-q)\times p} & 0_{(n-p-q)\times q} & I_{n-p-q} \end{pmatrix}$$

to both $\hat{A}$ and $\hat{B}$ yields $\tilde{A} := Q_3^T \hat{A} Q_3 = \hat{A} = \bar{A}$ and

$$\tilde{B} := Q_3^T \hat{B} Q_3 = \begin{pmatrix} B_1 - B_4 B_6^{-1} B_4^T & 0_{p\times q} & B_5 \\ 0_{q\times p} & B_6 & 0_{q\times(n-p-q)} \\ B_5^T & 0_{(n-p-q)\times q} & 0_{n-p-q} \end{pmatrix},$$

which take exactly the forms in (1) and (2), respectively.

In summary, letting $U := Q_1 Q_2 Q_3$ yields the forms of $U^T A U$ and $U^T B U$ given in (1) and (2) in the lemma, respectively. ∎

We present next a theorem which extends the results in [21] to situations where both matrices are singular.

THEOREM 6. *Two singular matrices $A$ and $B$, which take the forms* (1) *and* (2), *respectively, are SD if and only if the Jordan normal form of $A_1^{-1}\mathcal{B}_1$ is diagonal and $\mathcal{B}_2$ is a zero matrix or $r = 0$ ($\mathcal{B}_2$ does not exist).*

*Proof.* Recall that $A$ and $B$ are SD if and only if $\tilde{A}$ and $\tilde{B}$ are SD. (The notations $\tilde{A}$ and $\tilde{B}$ are the same as those in Lemma 5.) We can always choose a sufficiently large $\lambda$ such that the first $p$ columns of $\tilde{B}+\lambda\tilde{A}$ are linearly independent. For example, we can set $\lambda = \max_{i=1,\dots,p}\sum_{j=1}^{p}|b_{ij}|/|a_{ii}|+1$, where $a_{ii}$ is the $i$th diagonal entry of the nonsingular diagonal matrix $A_1$ and $b_{ij}$ is the $(i,j)$th entry of $\mathcal{B}_1$, and, by noting that $A_1$ is diagonal, $\lambda A_1+\mathcal{B}_1$ becomes diagonally dominant and thus nonsingular.

If $\mathcal{B}_2$ does not exist or is a zero matrix, $\tilde{A}$ and $\tilde{B}$ are SD $\Leftrightarrow$ $\tilde{B}+\lambda\tilde{A}$ and $\tilde{A}$ are SD $(\lambda>0)$ $\Leftrightarrow$ $\lambda A_1+\mathcal{B}_1$ and $A_1$ are SD $\Leftrightarrow$ $\mathcal{B}_1$ and $A_1$ are SD $\Leftrightarrow$ the Jordan normal form of $A_1^{-1}\mathcal{B}_1$ is diagonal, where the second "$\Leftrightarrow$" follows Lemma 4.

If the columns of $\mathcal{B}_2$ are linearly independent, according to Lemma 3, $\tilde{A}$ and $\lambda\tilde{A}+\tilde{B}$ are not SD $\Rightarrow$ $\tilde{A}$ and $\tilde{B}$ are not SD $\Rightarrow$ $A$ and $B$ are not SD.

If the columns of $\mathcal{B}_2$ are linearly dependent, we can find a nonsingular congruent matrix

$$Q_4 := \begin{pmatrix} I_p & 0_{p\times q} & 0_{p\times(n-p-q)} \\ 0_{q\times p} & I_q & 0_{q\times(n-p-q)} \\ 0_{(n-p-q)\times p} & 0_{(n-p-q)\times q} & T_{(n-p-q)\times(n-p-q)} \end{pmatrix}$$

such that $\breve{A} := Q_4^T\tilde{A}Q_4 = \tilde{A}$ and

$$\breve{B} := Q_4^T\tilde{B}Q_4 = \begin{pmatrix} \mathcal{B}_1 & 0_{p\times q} & \mathcal{B}_4 & 0_{p\times(n-p-q-s)} \\ 0_{q\times p} & \mathcal{B}_3 & 0_{q\times s} & 0_{q\times(n-p-q-s)} \\ \mathcal{B}_4^T & 0_{s\times q} & 0_s & 0_{s\times(n-p-q-s)} \\ 0_{(n-p-q-s)\times p} & 0_{(n-p-q-s)\times q} & 0_{(n-p-q-s)\times s} & 0_{n-p-q-s} \end{pmatrix},$$

where $T_{(n-p-q)\times(n-p-q)}$ is the product of elementary matrices which makes

$$\mathcal{B}_2T_{(n-p-q)\times(n-p-q)} = (\mathcal{B}_4, 0),$$

and $\mathcal{B}_4$ is a $p\times s$ $(s<n-p-q)$ matrix with full column rank. Then according to Lemma 3, similar to the case where $\mathcal{B}_2$ is of full column rank, $A$ and $B$ are not SD.

We complete the proof of the theorem. ∎

Algorithm 1 provides us an algorithmic procedure to verify whether two given symmetric matrices $A$ and $B$ are SD.

We next demonstrate the computational procedure to find the congruent matrix in checking whether a pair of matrices is SD via an illustrative example. The notations in the example follow those in Algorithm 1.

*Example* 1. Let

$$A := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$B := \begin{pmatrix} 1 & 2 & 0 & 0 & 3 & 0 \\ 2 & 5 & 1 & 0 & 0 & 0 \\ 0 & 1 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 0 \\ 3 & 0 & 0 & 2 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

**Algorithm 1** Procedure to check whether two matrices $A$ and $B$ are SD

**Input:** Two symmetric $n \times n$ matrices $A$ and $B$.

Apply the spectral decomposition to $A$ such that $\bar{A} := Q_1^T A Q_1 = \mathrm{diag}(A_1, 0)$, where $A_1$ is a nonsingular diagonal matrix, and express $\bar{B} := Q_1^T B Q_1 = \left(\begin{smallmatrix} B_1 & B_2 \\ B_2^T & B_3 \end{smallmatrix}\right)$

Apply the spectral decomposition to $B_3$ such that $V_1^T B_3 V_1 = \left(\begin{smallmatrix} B_6 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where $B_6$ is a nonsingular diagonal matrix; define $Q_2 := \mathrm{diag}(I, V_1)$ and set $\hat{A} := Q_2^T \bar{A} Q_2 = \bar{A}$ and

$$\hat{B} := Q_2^T \bar{B} Q_2 = \begin{pmatrix} B_1 & B_4 & B_5 \\ B_4^T & B_6 & 0 \\ B_5^T & 0 & 0 \end{pmatrix}$$

**if** $B_5$ exists and $B_5 \neq 0$ **then**

  **return** "not SD"

**else**

  Define

$$Q_3 := \begin{pmatrix} I_p & 0_{p\times q} & \\ -B_6^{-1}B_4^T & I_q & \\ & & I_{n-p-q} \end{pmatrix};$$

  further define $\tilde{A} := Q_3^T \hat{A} Q_3 = \hat{A} = \bar{A}$,

$$\tilde{B} := Q_3^T \hat{B} Q_3 = \begin{pmatrix} B_1 - B_4 B_6^{-1} B_4^T & 0 & \\ 0 & B_6 & \\ & & 0 \end{pmatrix}$$

  **if** the Jordan normal form of $A_1^{-1}(B_1 - B_4 B_6^{-1} B_4^T)$ is a diagonal matrix, i.e., $V_2^{-1} A_1^{-1}(B_1 - B_4 B_6^{-1} B_4^T) V_2 = \mathrm{diag}(\lambda_1 I_{n_1}, \ldots, \lambda_t I_{n_t})$, where $V_2$ is nonsingular **then**

    Find $R_k$, $k = 1, \ldots, t$, which is a spectral decomposition matrix of the $k$th diagonal block of $V_2^T A_1 V_2$ ; Define $R := \mathrm{diag}(R_1, \ldots, R_t)$, $Q_4 := \mathrm{diag}(V_2 R, I_{q+r})$, and $P := Q_1 Q_2 Q_3 Q_4$

    **return** two diagonal matrices $Q_4^T \tilde{A} Q_4$ and $Q_4^T \tilde{B} Q_4$ and the corresponding congruent matrix $P$

  **else**

    **return** "not SD"

  **end if**

**end if**

Applying the following congruent matrix which is derived in the appendix,

$$P := \begin{pmatrix} 3.98 & 194.23 & -0.21 & 0 & 0 & 0 \\ 7.09 & -27.55 & -0.29 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 3.98 & 194.22 & -0.21 & -0.45 & -0.89 & 0 \\ -3.98 & -194.22 & 0.21 & -0.89 & 0.45 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

we obtain two diagonal matrices,

$$P^T A P = \mathrm{diag}(226, 40772, 9, 0, 0, 0)$$

and

$$P^T BP = \mathrm{diag}(354, -93111, 7, 6, 1, 0).$$

*Remark* 7. Computational methods for obtaining the Jordan normal form that are of complexity $O(n^3)$ can be found in [1] and [5]. One computational problem in computing the SD form for two matrices is the instability of the numerical methods for the Jordan normal form; see [4, Chapter 7], [5], and [8]. As a small perturbation in a problem setting may cause a significant change in the formation of the Jordan blocks, it becomes difficult to deal with large-scale problems; see [1]. Our situation, however, is special, as we only need to calculate the Jordan normal form for real eigenvalues and we stop when the block size of any Jordan block is larger than 1 (implying that the two matrices are not SD). On the other hand, several symbolic computational methods have been developed to calculate the exact Jordan normal form (please refer to [11] and [16]).

**2.2. Applications to the generalized trust region subproblem and its variants.** Recall that the TRS is a special QCQP problem with one quadratic constraint:

$$\text{(TRS)} \quad \begin{aligned} &\min \frac{1}{2}x^T Bx + a^T x \\ &\text{s.t. } \|x\|_2 \le 1, \end{aligned}$$

where $B$ is an $n \times n$ symmetric matrix and $a \in \Re^n$. Numerous solution methods for problem formulation TRS have been developed in the literature; see, for example, [12] and [15].

We consider the following extension of TRS, which is called the generalized trust region subproblem (GTRS) in the literature:

$$\text{(GTRS)} \quad \begin{aligned} &\min \frac{1}{2}x^T Bx + a^T x \\ &\text{s.t. } \frac{1}{2}x^T Ax + c^T x + \frac{1}{2}d \le 0, \end{aligned}$$

where $A$ and $B \in S^n$ are $n \times n$ symmetric matrices, $a, c, x \in \Re^n$, and $d \in \Re$. Sturm and Zhang [19] prove the equivalence between the primal problem and its SDP relaxation for the GTRS if the Slater condition holds. However, the state of the art of SDP solvers does not support effective implementation in solving large-scale GTRSs. Several other methods have been proposed for solving large-scale instances of GTRSs under different conditions; see [2], [13], and [17].

The SD condition developed above in section 2.1 could find its immediate applications in the GTRS. When the two matrices $A$ and $B$ are SD, the problem is equivalent to the following SOCP problem as shown in [2]:

$$\begin{aligned} &\min \delta^T y + \epsilon^T x \\ &\text{s.t. } \alpha^T y + \beta^T x + \frac{1}{2}d \le 0, \\ &\qquad \frac{1}{2}x_i^2 \le y_i, \ i = 1, \ldots, n, \end{aligned}$$

where $\delta$, $\alpha$, $\epsilon$, $\beta \in \Re^n$, $\delta = \mathrm{Diag}(P^T BP)$, $\alpha = \mathrm{Diag}(P^T AP)$, $\epsilon = P^T a$, $\beta = P^T b$, and $P$ is the congruent matrix that makes both $P^T AP$ and $P^T BP$ diagonal. As SOCP reformulation can be solved much faster than SDP reformulation, it becomes possible to solve large-scale GTRSs using SOCP reformulation when a problem is identified to be SD. Algorithm 1 actually enables us to verify whether a given instance of the GTRS is equivalent to its SOCP relaxation. Note that if the inequality in the GTRS becomes an equality, the above analysis still holds true; see [2].

Problem TRS is always SD, as the Jordan normal form of $I^{-1}B = B$ is a diagonal matrix (due to the fact that the Jordan normal form of the symmetric matrix $B$ is diagonal). Thus, problem TRS is always equivalent to its SOCP relaxation.

When $B - \lambda A \succ 0$ holds for some $\lambda \in \Re$, this instance of GTRS is classified as a regular case [17]. It is clear that all regular cases are SD, since $A$ and $B$ under the above regularity condition are SD as discussed in section 1. We thus claim that any regular case of the GTRS has an equivalent SOCP relaxation. We can further conclude that the applicability of the SD approach is wider than the existing methods based on regularity conditions.

The SD condition can also be applied to solve the following interval constrained GTRS (IGTRS):

$$\text{(IGTRS)} \quad \min \frac{1}{2}x^T Bx + a^T x$$
$$\text{s.t. } l \leq \frac{1}{2}x^T Ax + c^T x \leq u,$$

where lower and upper bounds $l$ and $u \in \Re$ are chosen such that $-\infty < l < u < +\infty$. For regular cases of IGTRS, [14] extends the results in [17] to the IGTRS. Adopting Algorithm 1 developed in this paper to identify whether a given instance of IGTRS is SD, we can verify whether the problem can be reduced to an SOCP problem based on the results in [2] for QCQP with two quadratic constraints, which will be discussed in section 4. We emphasize that the regularity condition implies SD. Thus, the applicability of our approach is wider than the results in [14].

**3. Simultaneous diagonalization for finite number of matrices.** We now develop a method of checking simultaneous diagonalization for $m$ symmetric matrices $A_1, A_2, \ldots, A_m$, which have a semidefinite matrix pencil, i.e., there exists $\lambda \in \Re^m$ with $\lambda \neq 0$, such that $\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_m A_m \succeq 0$. Note that this condition is always assumed in the SDP relaxation for a QCQP problem; otherwise the SDP relaxation is unbounded from below; see [9, 18].

It is well known from [7] that if two matrices $A$ and $B$ commute, they are SD. It is interesting to extend this sufficient condition to a necessary and sufficient condition. The results in the following lemma have been mentioned in [2]. We provide its detailed proof in the following, which will be used to prove its extension, Theorem 9.

LEMMA 8. *Suppose that $A$ and $B$ are two symmetric $n \times n$ matrices, and $I$ is the $n \times n$ identity matrix. Then $A$, $B$, and $I$ are SD via an orthogonal congruent matrix if and only if $A$ and $B$ commute.*

*Proof.* The "⇒" part: If $A$, $B$, and $I$ are SD, then there exists a nonsingular matrix $P$ such that $P^T AP$, $P^T BP$, and $P^T IP$ are diagonal. Since $P^T IP$ is positive definite and diagonal, there exists a diagonal matrix $Q$ such that $Q^T P^T IPQ = I$. Let $U = PQ$; then $U^T AU$, $U^T BU$, and $U^T IU$ are all diagonal and $U^T IU = I$. So $U$ is an orthogonal matrix. Furthermore, $U^T AU$ and $U^T BU$ are diagonal ⇒ $U^T AUU^T BU = U^T BUU^T AU \Rightarrow AB = BA$, i.e., $A$ and $B$ commute.

The "⇐" part: Let $P$ be the spectral decomposition matrix for $A$ such that $P^T P = I$ and $P^T AP$ is diagonal. If $A$ and $B$ commute, i.e., $AB = BA$, then by $PP^T = I$ we obtain $P^T APP^T BP = P^T BPP^T AP$. We can always assume $P$ to be a nonsingular matrix such that $P^TAP = \mathrm{diag}(\lambda_1 I_{n_1}, \lambda_2 I_{n_2}, \dots, \lambda_k I_{n_k})$, where $k \le n$, $n_k$ is the algebraic multiplicity of $\lambda_i$. So, $P^TAPP^TBP = P^TBPP^TAP \Rightarrow P^TBP = \mathrm{diag}(B_1, B_2, \dots, B_k)$ is a block diagonal matrix and $\dim(B_i) = \dim(I_{n_i})$ $\forall 1 \le i \le k$. Applying the spectral decomposition to all $B_i$, we obtain $k$ diagonal matrices $Q_i^T B_i Q_i$, where $Q_i^T Q_i = I_{n_i}$, and $\dim(Q_i) = \dim(I_{n_i})$ $\forall 1 \le i \le k$. By denoting $Q = \mathrm{diag}(Q_1, Q_2, \dots, Q_k)$, $Q^TP^TBPQ$ and $Q^TP^TAPQ$ are both diagonal matrices. In addition, $Q^TP^TIPQ = I$. Thus, $A$, $B$, and $I$ are SD. ∎

One byproduct of Lemma 8 is its applicability to solving the following Celis–Dennis–Tapia (CDT) problem [3]:

$$\begin{aligned} &\min \frac{1}{2}x^TBx + a^Tx \\ &\text{s.t. } \| x \|_2 \le 1, \\ &\qquad \| A^Tx + c \|_2 \le 1. \end{aligned}$$

Note that we can view the matrix in the quadratic term of the first inequality in the above CDT problem as $I$. We can then use Lemma 8 to identify if the three matrices are SD, i.e., if $B$ and $AA^T$ commute. If so, we can then apply the method in [2] to solve the problem if one of the KKT multipliers is 0.

The following theorem shows the SD condition for the identity matrix $I$ and other $m$ symmetric matrices when the congruent matrix is orthogonal, which is a classical result in linear algebra; see [7]. For the sake of completeness, we give a brief proof here.

THEOREM 9. *Matrices $I, A_1, A_2, \dots, A_m$ are SD if and only if $A_i$ commutes with $A_j$ $\forall i, j = 1, 2, \dots, m$, $i \ne j$.*

*Proof.* Note that $I, A_1, A_2, \dots, A_m$ are SD if and only if there exists a matrix $P$ such that $P^TP = I$, and $P^TA_1P, \dots, P^TA_mP$ are all diagonal matrices. Thus, $P$ is orthogonal.

We first prove the "⇒" part. Since there is an orthogonal matrix $P$, which makes $P^TA_iP$, $i = 1, \dots, m$, all diagonal, $P^TA_iPP^TA_jP = P^TA_jPP^TA_iP$. So $P^TA_iA_jP = P^TA_jA_iP$, and, thus, $A_iA_j = A_jA_i$, $i, j = 1, \dots, m$, $i \ne j$.

Next we prove the "⇐" part. As the case where $m = 2$ has already been proved in Lemma 8, we will use the induction principle to prove the general case. Suppose that $Q_1^TA_1Q_1$ is the spectral decomposition of $A_1$ such that $Q_1^TQ_1 = I$ and $Q_1^TA_1Q_1 = \mathrm{diag}(\lambda_1 I_{n_1}, \lambda_2 I_{n_2}, \dots, \lambda_k I_{n_k})$, with $1 \le k \le n$ and $\lambda_i \ne \lambda_j$, if $i \ne j$, $1 \le i, j \le k$. Then, $A_1A_i = A_iA_1 \Rightarrow Q_1^TA_1Q_1Q_1^TA_iQ_1 = Q_1^TA_iQ_1Q_1^TA_1Q_1 \Rightarrow \tilde{A}_i := Q_1^TA_iQ_1$ is a block diagonal matrix in the form of $\mathrm{diag}(\tilde{A}_{i_1}, \tilde{A}_{i_2}, \dots, \tilde{A}_{i_k})$, with $\dim(\tilde{A}_{i_l}) = \dim(I_{n_l})$, $i = 1, \dots, m$, $l = 1, 2, \dots, k$. Furthermore,

$$A_iA_j = A_jA_i \Rightarrow Q_1^TA_iQ_1Q_1^TA_jQ_1 = Q_1^TA_jQ_1Q_1^TA_iQ_1.$$

So $\tilde{A}_{i_l}$ commutes with $\tilde{A}_{j_l}$ $\forall 2 \le i, j \le m$, $l = 1, 2, \dots, k$. Now for every $l = 1, 2, \dots, k$, it remains to prove that if $m - 1$ matrices $\tilde{A}_{i_l}$, $2 \le i \le m$, commute with each other, then they are SD via some orthogonal matrix $R_l$. According to the induction principle and Lemma 8, which represents the case $m = 2$, we complete the proof. ∎

Theorem 9 also gives a recursion procedure to identify if $I, A_1, A_2, \dots, A_m$ are SD.

By applying Theorem 9, we have the following theorem under a definite matrix pencil condition.

THEOREM 10. *If there exists $\lambda \in \Re^m$ such that $\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_m A_m \succ 0$, where, without loss of generality, $\lambda_m$ is assumed not to be zero, then $A_1, A_2, \ldots, A_m$ are SD if and only if $P^T A_i P$ commute with $P^T A_j P$ $\forall i \neq j$, $1 \leq i,\ j \leq m-1$, where $P$ is any nonsingular matrix that makes $P^T(\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_m A_m)P = I$.*

*Proof.* The matrix $\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_m A_m$ is positive definite and thus there exists an orthogonal matrix $U$ such that $D = U^T(\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_m A_m)U$ is a diagonal matrix. Define $D^{\frac{1}{2}} = \mathrm{diag}(D_{11}^{\frac{1}{2}}, D_{22}^{\frac{1}{2}}, \ldots, D_{nn}^{\frac{1}{2}})$, and then $D^{-\frac{1}{2}} U^T(\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_m A_m)UD^{-\frac{1}{2}} = I$. Define $P = UD^{-\frac{1}{2}}$. We then have $P^T(\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_m A_m)P = I$. Note that $A_1, \ldots, A_{m-1}, \lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_m A_m$ are SD if and only if $P^T A_1 P, \ldots, P^T A_{m-1} P, P^T(\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_m A_m)P$ are SD. From Theorem 9, we conclude that $P^T A_1 P, \ldots, P^T A_{m-1} P$, $P^T(\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_m A_m)P$ are SD if and only if $P^T A_i P$ commutes with $P^T A_j P$ $\forall i \neq j$, $1 \leq i,\ j \leq m-1$. So we only need to prove that $A_1, A_2, \ldots, A_m$ are SD $\Leftrightarrow$ $A_1, A_2, \ldots, A_{m-1}, \lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_m A_m$ are SD.

We first prove the "$\Rightarrow$" part: If $A_1, A_2, \ldots, A_m$ are SD, then there exists a nonsingular matrix $P$ such that $P^T A_1 P,\ P^T A_2 P, \ldots, P^T A_m P$ are all diagonal matrices. Thus, $\lambda_1 P^T A_1 P + \lambda_2 P^T A_2 P + \cdots + \lambda_m P^T A_m P$ is a diagonal matrix. We therefore obtain that $A_1, A_2, \ldots, A_{m-1}, \lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_m A_m$ are SD.

We next prove the "$\Leftarrow$" part: If $A_1, A_2, \ldots, A_{m-1}, \lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_m A_m$ are SD, then there exists a nonsingular matrix $P$ such that $P^T A_1 P,\ P^T A_2 P, \ldots, P^T A_{m-1} P$, $P^T(\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_m A_m)P$ are all diagonal. Since $\lambda_m$ is not zero, $P^T A_m P = [P^T(\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_m A_m)P - \sum_{i=1}^{m-1} \lambda_i P^T A_i P]/\lambda_m$ is also diagonal. ∎

For the semidefinite matrix pencil case, we have the following results.

THEOREM 11. *Suppose that $A_1, A_2, \ldots, A_m$ and $B$ are $n \times n$ symmetric matrices with the forms*

$$A_i := \begin{pmatrix} A_i^1 & A_i^2 \\ (A_i^2)^T & A_i^3 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} I_p & \\ & 0 \end{pmatrix},$$

*where $\dim A_i^1 = \dim I_p = p < n$ $\forall 1 \leq i \leq m$, and $A_1^3$ is nonsingular. Then $A_1, A_2, \ldots, A_m$ and $B$ are SD if and only if the following three conditions are satisfied:*
1. *$A_i^3$ are SD, $i = 1, \ldots, m$.*
2. *$A_i^2 = A_1^2 (A_1^3)^{-1} A_i^3$, $i = 2, \ldots, m$.*
3. *$A_i^1 - A_i^2 (A_1^3)^{-1} (A_1^2)^T$, $i = 1, \ldots, m$, mutually commute.*

*Proof.* The "$\Rightarrow$" part: If $A_1, A_2, \ldots, A_m$ and $B$ are SD, there exists a nonsingular matrix $P$ such that $P^T A_1 P, P^T A_2 P, \ldots, P^T A_m P$ and $P^T B P$ are all diagonal, and we can assume, without loss of generality, $P^T B P = B$. As $P^T B P = B$ if and only if

$$\begin{pmatrix} P_1^T & P_3^T \\ P_2^T & P_4^T \end{pmatrix} \begin{pmatrix} I_p & \\ & 0 \end{pmatrix} \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} = \begin{pmatrix} P_1^T P_1 & P_1^T P_2 \\ P_2^T P_1 & P_2^T P_2 \end{pmatrix} = \begin{pmatrix} I_p & \\ & 0 \end{pmatrix},$$

we conclude that $P_1$ is orthogonal and $P_2 = 0$. The nonsingularity of $P$ further implies that $P_4$ is nonsingular.

Furthermore, we have

$$\begin{aligned} P^T A_i P &= \begin{pmatrix} P_1^T & P_3^T \\ P_2^T & P_4^T \end{pmatrix} \begin{pmatrix} A_i^1 & A_i^2 \\ (A_i^2)^T & A_i^3 \end{pmatrix} \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \\ &= \begin{pmatrix} P_1^T A_i^1 P_1 + P_3^T A_i^3 P_3 + P_1^T A_i^2 P_3 + P_3^T (A_i^2)^T P_1 & P_1^T A_i^2 P_4 + P_3^T A_i^3 P_4 \\ P_4^T (A_i^2)^T P_1 + P_4^T A_i^3 P_3 & P_4^T A_i^3 P_4 \end{pmatrix}. \end{aligned}$$

Since $P^TA_iP,\ i=1,\dots,m$, are diagonal, the lower right blocks $P_4^TA_i^3P_4$ are diagonal, $i=1,\dots,m$. Together with the fact that $P_4$ is nonsingular, we conclude that $A_i^3$ are SD, $i=1,\dots,m$.

Moreover, because $P^TA_iP,\ i=1,\dots,m$, are diagonal, we have $P_4^T(A_i^2)^TP_1+P_4^TA_i^3P_3=0$ for $i=1,\dots,m$. As $P_4$ is nonsingular, we have $(A_i^2)^TP_1+A_i^3P_3=0$ for $i=1,\dots,m$. Solving $(A_1^2)^TP_1+A_1^3P_3=0$ yields $P_3=-(A_1^3)^{-1}(A_1^2)^TP_1$. Substituting $P_3=-(A_1^3)^{-1}(A_1^2)^TP_1$ into $(A_i^2)^TP_1+A_i^3P_3=0$ and by the nonsingularity of $P_1$, we further have $A_i^3(A_1^3)^{-1}(A_1^2)^T=(A_i^2)^T$, $i=2,\dots,m$.

Substituting $P_3=-(A_1^3)^{-1}(A_1^2)^TP_1$ into the (1,1)th block of $P^TA_iP$, $P_1^TA_i^1P_1+P_3^TA_i^3P_3+P_1^TA_i^2P_3+P_3^T(A_i^2)^TP_1$ gives rise to $P_1^T(A_1^1-A_1^2(A_1^3)^{-1}(A_1^2)^T)P_1$ when $i=1$, and

$$\begin{aligned}&P_1^T(A_i^1+A_1^2(A_1^3)^{-1}A_i^3(A_1^3)^{-1}(A_1^2)^T-A_i^2(A_1^3)^{-1}(A_1^2)^T-A_1^2(A_1^3)^{-1}(A_i^2)^T)P_1\\&=P_1^T(A_i^1-A_i^2(A_1^3)^{-1}(A_1^2)^T)P_1\end{aligned}$$

for $i=2,\dots,m$. Since the (1,1)th blocks of $P^TA_iP,\ i=1,\dots,m$, and $B$ are all diagonal matrices and $P_1$ is orthogonal, we conclude that $A_i^1-A_i^2(A_1^3)^{-1}(A_1^2)^T$, $i=1,2,\dots,m$, mutually commute.

The "$\Leftarrow$" part: If conditions 1, 2, and 3 hold, then $P^TA_iP,\ i=1,\dots,m$, and $P^TBP$ are all diagonal by choosing

$$P:=\begin{pmatrix}P_1 & 0\\ -(A_1^3)^{-1}(A_1^2)^TP_1 & P_4\end{pmatrix},$$

where $P_1$ is the orthogonal matrix such that $P_1^T(A_i^1-A_i^2(A_1^3)^{-1}(A_1^2)^T)P_1$, $i=1,\dots,m$, are all diagonal and $P_4$ is a nonsingular matrix such that $P_4^TA_i^3P_4$ are all diagonal, $i=1,\dots,m$. Note that the existence of $P_1$ is due to condition 3 and Theorem 9. $\square$

Now we extend the result in Theorem 11, in which $\lambda_1A_1+\lambda_2A_2+\cdots+\lambda_mA_m\succ 0$ holds true, to situations where only the following assumption holds.

ASSUMPTION 1. *For $m$ symmetric $n\times n$ matrices, $A_1,A_2,\dots,A_m$, there exists $\lambda\in\Re^m$ with $\lambda\neq 0$, such that $\lambda_1A_1+\lambda_2A_2+\cdots+\lambda_mA_m\succeq 0$, but there does not exist $\lambda\in R^m$ such that $\lambda_1A_1+\lambda_2A_2+\cdots+\lambda_mA_m\succ 0$. Without loss of generality, we assume $\lambda_m\neq 0$.*

For a given set of $A_i,\ i=1,\dots,m$, we can always use an SDP formulation to verify whether there exists $\lambda\neq 0$, which satisfies Assumption 1. If the following SDP is feasible, then there exists such a $\lambda\neq 0$ because otherwise $tr(\sum_{i=1}^m\lambda_iA_i)=0$,

$$\begin{aligned}&\min 0\\ &\text{s.t.} \sum_{i=1}^m\lambda_iA_i\succeq 0,\\ &\qquad tr\left(\sum_{i=1}^m\lambda_iA_i\right)\geq 1.\end{aligned}$$

When Assumption 1 holds, we can find a nonsingular matrix $Q_1$ and the corresponding $\lambda\in\Re^m$ such that $\mathcal{A}_m:=Q_1^T(\lambda_1A_1+\lambda_2A_2+\cdots+\lambda_mA_m)Q_1=\begin{pmatrix}I_p & \\ & 0\end{pmatrix}$, and

$$\mathcal{A}_i:=Q_1^TA_iQ_1=\begin{pmatrix}\mathcal{A}_i^1 & \mathcal{A}_i^2\\ (\mathcal{A}_i^2)^T & \mathcal{A}_i^3\end{pmatrix},\ i=1,2,\dots,m-1, \tag{3}$$

where $\dim \mathcal{A}_i^1 = \dim I_p = p < n$.

Furthermore, if all $\mathcal{A}_i^3$, $i = 1, \ldots, m$, are SD, then, by rearranging the common 0's to the lower right corner of the matrix, there exists a nonsingular matrix $Q_2 := \operatorname{diag}(I_p, V)$ such that

$$\mathbf{A}_m := Q_2^T \mathcal{A}_m Q_2 = \begin{pmatrix} I_p & \\ & 0 \end{pmatrix}, \tag{4}$$

and

$$\mathbf{A}_i := Q_2^T \mathcal{A}_i Q_2 = \begin{pmatrix} \mathbf{A}_i^1 & \mathbf{A}_i^2 & \mathbf{A}_i^4 \\ (\mathbf{A}_i^2)^T & \mathbf{A}_i^3 & 0 \\ (\mathbf{A}_i^4)^T & 0 & 0 \end{pmatrix}, \tag{5}$$

where $\mathbf{A}_i^1 = \mathcal{A}_i^1$, $\mathbf{A}_i^3$, $i = 1, 2, \ldots, m-1$, are all diagonal matrices and do not have common 0's in the same positions.

For any diagonal matrices $D$ and $E$, define $\operatorname{supp}(D) := \{i \mid D_{ii} \neq 0\}$ and $\operatorname{supp}(D) \cup \operatorname{supp}(E) := \{i \mid D_{ii} \neq 0 \text{ or } E_{ii} \neq 0\}$.

LEMMA 12. *For $k$ ($k \geq 2$) $n \times n$ nonzero diagonal matrices $D^1, D^2, \ldots, D^k$, if there exists no common 0's in the same position, then the following procedure will find $\mu_i \in \Re$, $i = 1, \ldots, k$, such that $\sum_{i=1}^k \mu_i D^i$ is nonsingular.*

*Step 1. Let $D = D^1$, $\mu_1 = 1$, and $\mu_i = 0$ for $i = 2, 3, \ldots, n$, $j = 1$.*

*Step 2. Let $D^* = D + \mu_{j+1} D^{j+1}$, where $\mu_{j+1} = \frac{s}{n}$, $s \in \{0, 1, \ldots, n\}$, with $s$ being chosen such that $D^* = D + \mu_{j+1} D^{j+1}$ and $\operatorname{supp}(D^*) = \operatorname{supp}(D) \cup \operatorname{supp}(D^{j+1})$.*

*Step 3. Let $D = D^*$, $j = j + 1$; if $D$ is nonsingular or $j = n$, STOP and output $D$; else, go to Step 2.*

*Proof.* We only need to prove that in Step 2 we can find a suitable $s$. The cardinality of $\operatorname{supp}(D^{j+1})$ is at most $n$ and that of $\operatorname{supp}(D)$ is at most $n-1$ (or we have already found a nonsingular matrix $D$). So if $\operatorname{supp}(D^{j+1}) = \operatorname{supp}(D)$, let $s = 0$, and we have $\operatorname{supp}(D^*) = \operatorname{supp}(D) \cup \operatorname{supp}(D^{j+1})$. Otherwise, for any $i \in \operatorname{supp}(D^{j+1})$, varying $s$ from 0 to $n$ will generate $n+1$ different values for $D_{ii} + \frac{s}{n} D_{ii}^{j+1}$, and at most one $s$ can lead to $D_{ii} + \frac{s}{n} D_{ii}^{j+1} = 0$. But as the cardinality of $\operatorname{supp}(D^{j+1}) \leq n$, we can always find an $s$ such that $D_{ii} + \frac{s}{n} D_{ii}^{j+1} \neq 0 \ \forall i \in \operatorname{supp}(D) \cup \operatorname{supp}(D^{j+1})$. One way to find such an $s$ is to test $s$ from 0 to $n$ to find out the value of $s$ such that $\operatorname{supp}(D + \frac{s}{n} D^{j+1}) = \operatorname{supp}(D) \cup \operatorname{supp}(D^{j+1})$. ∎

Let us define

$$\mathbf{D} := \sum_{i=1}^{m-1} \mu_i \mathbf{A}_i = \begin{pmatrix} \mathbf{D}_1 & \mathbf{D}_2 & \mathbf{D}_4 \\ (\mathbf{D}_2)^T & \mathbf{D}_3 & 0 \\ (\mathbf{D}_4)^T & 0 & 0 \end{pmatrix}, \tag{6}$$

where $\mu_i$, $i = 1, 2, \ldots, m-1$, are chosen, via the procedure in Lemma 12, such that $\mathbf{D}_3$ is nonsingular. Then we have the following theorem under the existence of a semidefinite matrix pencil.

THEOREM 13. *Under Assumption 1, $A_1, A_2, \ldots, A_m$ are SD $\Leftrightarrow$ $A_1, A_2, \ldots, A_{m-1}$ and $\tilde{A}_m := \lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_m A_m \succeq 0$ are SD $\Leftrightarrow$ $\mathcal{A}_i^3$ (defined in (3)), $i = 1, 2, \ldots, m$, are SD, and the following conditions are also satisfied:*
1. *$\mathbf{D}_4 = 0$ and $\mathbf{A}_i^4 = 0$, $i = 1, 2, \ldots, m-1$,*
2. *$\mathbf{A}_i^2 = \mathbf{D}_2 \mathbf{D}_3^{-1} \mathbf{A}_i^3$, $i = 1, 2, \ldots, m-1$,*
3. *$\mathbf{A}_i^1 - \mathbf{A}_i^2 \mathbf{D}_3^{-1} \mathbf{D}_2^T$, $i = 1, 2, \ldots, m-1$, mutually commute,*

*where $\mathbf{A}_i^1$, $\mathbf{A}_i^2$, $\mathbf{A}_i^3$, and $\mathbf{A}_i^4$ are defined in (5), and $\mathbf{D}$ is defined in (6).*

*Proof.* The first "⇔" can be proved by the same technique with the proof of Theorem 10.

We next prove the second "⇔."

The "⇒" part: $A_1, A_2, \ldots, A_{m-1}$ and $\tilde{A}_m$ are SD ⇒ $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ are SD (note that $\mathcal{A}_m = Q_1^T \tilde{A}_m Q_1$ by definition). Theorem 11 further shows that $\mathcal{A}_i^3$, $i = 1, 2, \ldots, m$, are SD. Then from (3), (4), and (5), we know that there exists a nonsingular matrix $Q_2$ such that $Q_2^T \mathcal{A}_i Q_2$, $i = 1, 2, \ldots, m$, are SD and have the forms in (4) and (5), i.e.,

$$\mathbf{A}_m := Q_2^T \mathcal{A}_m Q_2 = \begin{pmatrix} I_p & \\ & 0 \end{pmatrix}$$

and

$$\mathbf{A}_i := Q_2^T \mathcal{A}_i Q_2 = \begin{pmatrix} \mathbf{A}_i^1 & \mathbf{A}_i^2 & \mathbf{A}_i^4 \\ (\mathbf{A}_i^2)^T & \mathbf{A}_i^3 & 0 \\ (\mathbf{A}_i^4)^T & 0 & 0 \end{pmatrix}.$$

Then $\mathcal{A}_i$ and $\mathcal{A}_m$ are SD, $i = 1, 2, \ldots, m-1$ ⇔ $\mathbf{A}_i$ and $\mathbf{A}_m$ are SD, $i = 1, 2, \ldots, m-1$ ⇔ $\mathbf{D}$, $\mathbf{A}_i$, $i = 2, 3, \ldots, m-1$, and $\mathbf{A}_m$ are SD. Using the arguments in the proof of Lemma 5, we can transform $\mathbf{A}_m$ and $\mathbf{A}_i$ ($i = 1, \ldots, m-1$) to the forms in (1) and (2) with $\mathcal{B}_2 = \mathbf{A}_i^4$ ($i = 1, \ldots, m-1$), respectively. Thus, from Theorem 6, we can claim $\mathbf{A}_i^4 = 0$, $i = 1, 2, \ldots, m-1$, and $\mathbf{D}_4 = 0$. In addition, from Theorem 11, we can conclude that conditions 2 and 3 are also satisfied.

The "⇐" part: Note that $Q_1^T A_i Q_1 = \mathcal{A}_i$, $i = 1, \ldots, m-1$. Then if $\mathcal{A}_i^3$ are SD, $i = 1, 2, \ldots, m$, there exists a nonsingular matrix $Q_2$ such that (4) and (5) hold. Further, define

$$P := \begin{pmatrix} P_1 & 0 & 0 \\ -(\mathbf{D}_3)^{-1}(\mathbf{D}_2)^T P_1 & I_q & 0 \\ 0 & 0 & I_{n-p-q} \end{pmatrix}, \tag{7}$$

where $P_1$ is an orthogonal matrix such that $P_1^T(\mathbf{A}_i^1 - \mathbf{A}_i^2(\mathbf{D}_3)^{-1}(\mathbf{D}_2)^T)P_1$, $i = 1, \ldots, m-1$, are all diagonal, and $q = \dim(\mathbf{A}_1^3)$. Then one can check that

$$\begin{aligned} & P^T \mathbf{A}_i P \\ &= \begin{pmatrix} \begin{matrix} P_1^T(\mathbf{A}_i^1 - \mathbf{A}_i^2 \mathbf{D}_3^{-1} \mathbf{D}_2^T - \mathbf{D}_2 \mathbf{D}_3^{-1} (\mathbf{A}_i^2)^T \\ + \mathbf{D}_2 \mathbf{D}_3^{-1} \mathbf{A}_i^3 \mathbf{D}_3^{-1} \mathbf{D}_2^T) P_1 \end{matrix} & P_1^T(\mathbf{A}_i^2 - \mathbf{D}_2 \mathbf{D}_3^{-1} \mathbf{A}_i^3) & 0 \\ ((\mathbf{A}_i^2)^T - \mathbf{A}_i^3 \mathbf{D}_3^{-1} \mathbf{D}_2^T) P_1 & \mathbf{A}_i^3 & 0 \\ 0 & 0 & 0_{n-p-q} \end{pmatrix} \\ &= \begin{pmatrix} P_1^T(\mathbf{A}_i^1 - \mathbf{A}_i^2 \mathbf{D}_3^{-1} \mathbf{D}_2^T) P_1 & & \\ & \mathbf{A}_i^3 & \\ & & 0_{n-p-q} \end{pmatrix}, \end{aligned}$$

$i = 1, \ldots, m-1$, and $P^T \mathbf{A}_m P = \mathrm{diag}(I_p, 0)$ are all diagonal. Thus, we have that $A_1, A_2, \ldots, A_{m-1}$ and $\tilde{A}_m := \lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_m A_m \succeq 0$ are SD via the congruent matrix $Q_1 Q_2 P$. ∎

Although we choose $\mu$ following the procedure in Lemma 12, the proof shows that Theorem 13 always holds for every $\mu$ such that $\mathbf{D}_3$ is nonsingular. Based on the above discussion, we develop a systematic procedure in Algorithm 2 to verify whether the given $m$ matrices are SD.

**Algorithm 2** Procedure to check whether $m$ matrices are SD

**Input:** $m$ symmetric matrices $A_1, A_2, \ldots, A_m$, which satisfy that $\exists \lambda \in \Re^m,\ \lambda \neq 0$, such that $\sum_{i=1}^m \lambda_i A_i \succeq 0$

Find a nonsingular matrix $Q_1$ such that $\mathcal{A}_m = Q_1^T(\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_m A_m)Q_1 = \begin{pmatrix} I_p & \\ & 0 \end{pmatrix}$, in which we assume $\lambda_m \neq 0$; otherwise we can exchange the index $m$ with $k$ for some $\lambda_k \neq 0$; Define $\mathcal{A}_i = Q_1^T A_i Q_1 = \begin{pmatrix} \mathcal{A}_i^1 & \mathcal{A}_i^2 \\ (\mathcal{A}_i^2)^T & \mathcal{A}_i^3 \end{pmatrix},\ i = 1, 2, \ldots, m-1$

**if** $\mathcal{A}_i^3,\ i = 1, 2, \ldots, m-1$, are not SD **then**

  **return** "not SD"

**else**

  Find a congruent matrix $V$ that makes all $\mathcal{A}_i^3$ diagonal,[1] and $Q_2 = \mathrm{diag}(I_p, V)$ that makes

$$Q_2^T Q_1^T \mathcal{A}_i Q_1 Q_2 = \begin{pmatrix} \mathbf{A}_i^1 & \mathbf{A}_i^2 & \mathbf{A}_i^4 \\ (\mathbf{A}_i^2)^T & \mathbf{A}_i^3 & 0 \\ (\mathbf{A}_i^4)^T & 0 & 0 \end{pmatrix},$$

  $i = 1, \ldots, m-1$

  Find $\mu_i,\ i = 1, 2, \ldots, m-1$, via the procedure in Lemma 12, such that $\mathbf{D}_3 := \sum_{i=1}^{m-1} \mu_i \mathbf{A}_i^3$ is nonsingular, and define $\mathbf{D}_2 := \sum_{i=1}^{m-1} \mu_i \mathbf{A}_i^2$

  **if** (i) $\mathbf{A}_i^4 = 0,\ i = 1, 2, \ldots, m-1$, (ii) $\mathbf{A}_i^2 = \mathbf{D}_2 \mathbf{D}_3^{-1} \mathbf{A}_i^3, i = 2, \ldots, m-1$, and (iii) $\mathbf{A}_i^1 - \mathbf{A}_i^2 \mathbf{D}_3^{-1} \mathbf{D}_2^T,\ i = 2, \ldots, m-1$, mutually commute **then**

    **return** $U^T A_i U, i = 1, \ldots, m$, and $U = Q_1 Q_2 P$, where $P$ is defined in (7)

  **else**

    **return** "not SD"

  **end if**

**end if**

*Example* 2. The following example shows a case in which three matrices do not mutually commute, but they are SD. Consider $A = \begin{pmatrix} 1 & 2 \\ 2 & 20 \end{pmatrix}$, $B = \begin{pmatrix} -1 & -2 \\ -2 & -28 \end{pmatrix}$, and $C = \begin{pmatrix} 3 & 6 \\ 6 & -20 \end{pmatrix}$. It is easy to check that $AB \neq BA$. Let $U = \begin{pmatrix} 1 & -0.5 \\ 0 & 0.25 \end{pmatrix}$. Then, we can show that $A$, $B$, and $C$ are SD, as $U^T AU = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $U^T BU = \begin{pmatrix} -1 & 0 \\ 0 & -1.5 \end{pmatrix}$, and $U^T CU = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$.

**4. Applications to quadratically constrained quadratic programming.** Consider the following QCQP problem:

$$\text{(QP)} \quad \min f^0(x) \quad \text{s.t. } f^i(x) \leq 0,\ i = 1, \ldots, m,$$

where $f^0(x) = \frac{1}{2}x^T A_0 x + a_0^T x$, and $f^i(x) = \frac{1}{2}x^T A_i x + a_i^T x + \frac{1}{2}d_i,\ A_0, A_i \in S^n$, $a_0, a_i \in \Re^n$, and $d_i \in \Re$, $i = 1, \ldots, m$.

Let us consider first the homogeneous case where $a_i = 0,\ i = 0, 1, \ldots, m$. If $A_0, A_1, \ldots, A_m$ are SD with $\bar{A}_i$ being the diagonalized matrix of $A_i$, then (QP) is

[1]In the algorithm, we always rearrange the common 0's to the lower right corner of the $V^T \mathcal{A}_i^3 V$, $i = 1, \ldots, m-1$, via the congruent matrix $V$, and we use a recursion to find a congruent matrix $V$ that makes all $\mathcal{A}_i^3$'s diagonal.

equivalent to the following problem:

$$\text{(HQPSD)}\quad \min \sum_{k=1}^{n} \frac{1}{2}\text{Diag}(\bar{A}_0)_k x_k^2$$
$$\text{s.t.} \sum_{k=1}^{n} \frac{1}{2}\text{Diag}(\bar{A}_i)_k x_k^2 + \frac{1}{2}d_i \le 0,\ i = 1,\dots,m.$$

In fact, (HQPSD) is equivalent to the linear programming problem as follows:

$$\begin{aligned}\min &\sum_{k=1}^{n} \frac{1}{2}\text{Diag}(\bar{A}_0)_k y_k\\ \text{s.t.} &\sum_{k=1}^{n} \frac{1}{2}\text{Diag}(\bar{A}_i)_k y_k + \frac{1}{2}d_i \le 0,\ i = 1,\dots,m,\\ & y_k \ge 0,\ k = 1,\dots,n,\end{aligned}$$

which can be solved efficiently by several methods in the literature, e.g., the simplex algorithm.

Introducing $x_{n+1} = \pm 1$ in nonhomogeneous (QP) gives rise to the following equivalent problem formulation:

$$\text{(QP}'\text{)}\quad \begin{aligned}\min\ & \frac{1}{2}x^T A_0 x + a_0^T x x_{n+1}\\ \text{s.t.}\ & \frac{1}{2}x^T A_i x + a_i^T x x_{n+1} + \frac{1}{2}d_i x_{n+1}^2 \le 0,\ i = 1,\dots,m,\\ & x_{n+1}^2 = 1.\end{aligned}$$

Define $B_i = \begin{pmatrix} A_i & a_i \\ a_i^T & d_i \end{pmatrix}$, $i = 0, 1, \dots, m$ (here $d_0 = 0$), and $B_{m+1} = \begin{pmatrix} 0_n & \\ & 1 \end{pmatrix}$. We can rewrite (QP′) as the following homogeneous problem:

$$\begin{aligned}\min\ & \frac{1}{2}x^T B_0 x\\ \text{s.t.}\ & \frac{1}{2}x^T B_i x \le 0,\ i = 1,\dots,m,\\ & x^T B_{m+1} x = 1.\end{aligned}$$

We can then apply Algorithm 2 to check the SD condition of the above problem. In an SD situation, a nonhomogeneous QCQP problem can be also reduced to an equivalent linear programming formulation, similar to the homogeneous case.

Furthermore, when $m = 1$ or 2, the above problem has an exact relaxation in some cases, as shown below.

For $m = 1$, it is the GTRS which possesses an exact relaxation when the SD condition holds, as we discussed regarding TRS and GTRS in section 2.2.

For $m = 2$, we have the following SOCP relaxation when the three matrices are SD:

$$\text{(QP}_2\text{-SOCP)}\qquad \begin{aligned}\min\ & \delta^T y + \epsilon^T x\\ \text{s.t.}\ & \alpha^T y + \beta^T x + \frac{1}{2}d_1 \le 0,\\ & \eta^T y + \theta^T x + \frac{1}{2}d_2 \le 0,\\ & \frac{1}{2}x_i^2 \le y_i,\ i = 1,\dots,n,\end{aligned}$$

where $\delta, \epsilon, \alpha, \beta, \eta, \theta \in \Re^n$, $\delta = \text{Diag}(P^T A_0 P)$, $\alpha = \text{Diag}(P^T A_1 P)$, $\eta = \text{Diag}(P^T A_2 P)$, $\epsilon = P^T a_0$, $\beta = P^T a_1$, $\theta = P^T a_2$, and $P$ is the congruent matrix that makes $P^T A_i P$, $i = 0, 1, 2$, all diagonal. Ben-Tal and Hertog [2] demonstrate that if one of the KKT multipliers of the first two constraints in (QP$_2$−SOCP) is 0, then the SOCP relaxation is exact. For the SOCP relaxation of problem (IGTRS), the two quadratic constraints are relaxed to

$$\begin{aligned} &\alpha^T y + \beta^T x \le u, \\ &-\alpha^T y - \beta^T x \le -l, \end{aligned}$$

and one of the KKT multipliers must be 0, as the two boundary conditions cannot be binding simultaneously; i.e., it cannot be $l = \alpha^T y + \beta^T x = u$.

**5. Conclusion.** In this paper, we have succeeded in providing complete answers to the open question on simultaneous diagonalization (SD) posted in [6]. More specifically, we have identified a necessary and sufficient SD condition for any two real symmetric matrices and a necessary and sufficient SD condition for multiple (more than two) real symmetric matrices under the existence assumption of a semidefinite matrix pencil. Furthermore, we have demonstrated how SD can be utilized as a powerful instrument to verify the exactness of the SOCP relaxation of QCQP, especially with one or two quadratic constraints, and to facilitate the solution process. Our future work will include finding a necessary and sufficient SD condition for multiple matrices without the assumption of the semidefinite matrix pencil and finding more real-world applications of our SD procedure.

**Appendix A. Derivation of the congruent matrix for Example 1.** In this appendix, we show how to derive the congruent matrix $P$ for Example 1 by applying Algorithm 1: First note that $A$ is already in the form of $\text{diag}(\mathrm{A}_1, 0)$, and thus we do not need to go to line 1. Define

$$A_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix},$$

$$B_1 := \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 1 & 7 \end{pmatrix},$$

$$B_2 := \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$B_3 := \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Next we go to line 2. Applying the spectral decomposition to $B_3$ yields

$$V_1^T B_3 V_1 = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ where } V_1 := \begin{pmatrix} -0.4472 & -0.8944 & 0 \\ -0.8944 & 0.4472 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Letting $Q_2 := \mathrm{diag}(I_3, V_1)$ gives rise to

$$\hat{B} := Q_2^T B Q_2 = \begin{pmatrix} B_1 & B_4 & 0 \\ B_4^T & B_6 & 0 \\ 0^T & 0 & 0 \end{pmatrix},$$

where

$$B_4 := \begin{pmatrix} -2.6833 & 1.3416 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and $B_6 := \left(\begin{smallmatrix} 6 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Note that since $B_5 = 0$, $A$ and $B$ are SD, and we skip lines 3–5. Following line 6, letting

$$Q_3 := \begin{pmatrix} I_3 & & \\ -B_6^{-1}B_4^T & I_2 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.4472 & 0 & 0 & 1 & 0 & 0 \\ -1.3416 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

further yields

$$\begin{aligned} \tilde{B} &:= Q_3^T \hat{B} Q_3 \\ &= \begin{pmatrix} B_1 - B_4 B_6^{-1} B_4^T & & B_5 \\ & B_6 & \\ B_5^T & & \end{pmatrix} \\ &= \begin{pmatrix} -2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 5 & 1 & 0 & 0 & 0 \\ 0 & 1 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Then we go to line 7. With

$$B_1 - B_4 B_6^{-1} B_4^T = \begin{pmatrix} -2 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 1 & 7 \end{pmatrix}, \; A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix},$$

we obtain

$$A_1^{-1}(B_1 - B_4 B_6^{-1} B_4^T) = \begin{pmatrix} -2 & 2 & 0 \\ 0.5 & 1.25 & 0.25 \\ 0 & 0.1111 & 0.7778 \end{pmatrix}.$$

Applying Jordan decomposition gives rise to $J := V_2^{-1} A_1^{-1}(B_1 - B_4 B_6^{-1} B_4^T) V_2 = \mathrm{diag}(1.5657, -2.2837, 0.7458)$, where

$$V_2 := \begin{pmatrix} 3.98 & 194.23 & -0.21 \\ 7.09 & -27.55 & -0.29 \\ 1 & 1 & 1 \end{pmatrix}.$$

Thus, $A_1$ and $B_1 - B_4 B_6^{-1} B_4^T$ are SD, and with $V_2^T A_1 V_2 = \mathrm{diag}(226, 40772, 9)$ and $V_2^T(B_1 - B_4 B_6^{-1} B_4^T)V_2 = \mathrm{diag}(354, -93111, 7)$. Next we go to line 8. Define

$$Q_4 := \mathrm{diag}(V_2, I_3) = \begin{pmatrix} 3.98 & 194.23 & -0.21 & & & \\ 7.09 & -27.55 & -0.29 & & & \\ 1 & 1 & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}.$$

Note that in the last step, $V_2^T A_1 V_2$ is already diagonal, so we do not need to apply spectral decomposition to $V_2^T A_1 V_2$. Now by letting the congruent matrix $P := Q_2 Q_3 Q_4$, we have

$$P^T A P = \mathrm{diag}(226, 40772, 9, 0, 0, 0),$$

and

$$P^T B P = \mathrm{diag}(354, -93111, 7, 6, 1, 0).$$

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